UNIVERSITÀ DEGLI STUDI DI MILANO

MASTER THESIS

Exact Out-of-Equilibrium Dynamics in Classical Integrable Field Theories

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Abstract

We study the statistical properties of the Non-Linear Schrödinger Equation (NLSE) in one spatial dimension, in out-of-equilibrium protocols and in absence of dissipation, focusing on the thermodynamic limit and on states with extensive number of particles. On the theoretical side, integrable models exhibit exotic feautures because of the presence of an infinite set of conserved quantities, which strongly constrain the dynamics and offer a unique opportunity to derive exact analytical results for such a non-perturbative phenomenon as equilibration. Moreover, this extensive set of charges breaks down ergodicity and, being the system closed, leads to relaxation towards the so-called Generalized Gibbs Ensemble (GGE)[1]. Experimental advances in the realm of ultra-cold atom systems have boosted the theoretical interest in such special models[2]. In this setting, the out-of-equilibrium properties have been extensively probed in the framework of the famous *quantum quench*[3–5] and the Lieb-Liniger model (LL) has received particular attention. It describes a system of massive bosons with contact interactions and it is the quantized version of the NLSE. There are several reasons motivating our research. Firstly, the NLSE can be viewed as the semiclassical limit of the LL model, that is the regime of high occupation numbers and so it can provide new insights about its quantum counterpart, resulting in an array of results which might be amenable to experimental tests [6, 7]. Secondly, classical systems are faesible of extensive numerical simulations, which are not possible for the quantum LL. Thirdly, we have at our disposal many theoretical tools developed for the LL model which we can use in a rather economic way in order to access similar information of the NLSE, after suitable semiclassical limit[8]. Specifically, the purpose of this thesis is to study the classical counterpart of the homogeneous quantum quench, through the exploration of the relaxation properties of the system and the determination of the emergent steady state, once the initial conditions are given. This goal will be achieved by merging theoretical techniques, such as the Inverse Scattering Method and Bethe Ansatz, and numerical computations for the *transfer matrix*, the latter encoding all the relevant information of the steady state[9]. In particular, we give exact analytic expressions for the full counting statistics of the particle density and its moments. Our findings are valid for the steady state coming from arbitrary conditions. Besides out-of-equilibrium setups, these formulas can be applied to equilibrium thermal states as well.

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To Mum and Dad...

Contents

Introduction

Quantum physics has been, since its birth, an industry of astonishing experimental results. On the other side, theorists, provided an array of models based of very few and physical assumptions, but neverthless able to explain such an non-intuitive quantum mechanical behaviour of Nature. Few decades before, Boltzmann, alone, gave rise to the branch of theoretical physics studying a large number of degrees of freedom based on probabilistic methods. The Hydrogen and Helium atoms spectrum were fully described and predicted by the quantum theory but as happend in classical physics the three body problem represented soon an insormountable difficulty: the problem of many body quantum systems was knocking at the doors. Applying Boltzmann ideas to quantum systems showed up in a fruitful extension of thermodynamics to the microscopic world. New and spectacular phenomena were supposed, observed and predicted like the Bose-Einstein condensation (BEC). The basic assumption of equilibrium statistical mechanics (ESM) is the so-called *ergodic hypotesis*, now a theorem in many instances [10, 11]: basically, through the entire time evolution of the system, it densely visits each region of phase space. The hypotesis directly allows to postulate, according to Boltzmann [12], the a priori equal probability for each state of the system. The important point is that a generic system, quantum or classical, is described by the Gibbs distribution at equilibrium, $p(E) = e^{-\beta H}$: we call this distribution *thermal state*. Physically, the process of thermalization happens because of the presence of non linear terms in the equations describing the evolution, so that they induce non trivial scattering processes between particles (or interactions between degrees of freedom) such that there is a consequent mixing of modes, leading ultimately to thermalization: in weakly interacting models, this simply means energy is shared equally by each normal mode of the unperturbed Hamiltonian. In turn, this implies that the potential energy has been transformed into kinetic energy. We will see, that for field theories, the responsible player of utlimate thermalization is the laplacian term appearing in the equations of motion. Generally speaking, there are two main paradigms to describe a certain system: one can use a "fundamental" description and take into account every degree of freedom by which the system is made and eventually take the thermodynamic limit; the second approach is that of a field theory. The aformentioned descriptions are often interchangable and the choice depends on the system at hand but it is worth to say that there are phenomena that needs to be necessarily described by field theories: the most outstanding example is the Standard Model. However, switching between continuum and discrete models does not come without a price. Field theories suffer from UV divergences, a situation best represented by the UV catastrophe which led to the development of quantum mechanics. The Wilson renormalization group [13] has pushed forward our understanding of the divergences: it gave us a practical tool to cure them and to understand whether this is possible or not. Through a coarse grain procedure it is possible to sum up short wave length modes unimportant in low energy physics and incorporate the contribution in the parameters defining the model. Another important aspect of field theories is that

of symmetry. Indeed, modern theories are built up by lagrangians with an underlying symmetry group, which can be local, like for gravity, or global like for electrodynamics. The mechanism of spontaneous symmetry breaking [14] represents a cornerstone in this respect and permitted to explain the existence of massive particles and magnetics materials. Thus, it should clear that the field-theoretic paradigm has been fundamental in our description of Nature. Neverthless, it still deserves attention in many aspects as we want to show. Our present study is devoted to the out-of-equilibrium dynamics of *classical integrable field theories* and we will adopt the field-theoretic point of view for the most part of the thesis. Integrability is a fascinating subject and a rare property of a system: a complicated theory, like non-linear partial differential equation or a many-body quantum system, in special cases, turns out to be exactly solvable. This happens because, despite the appearence, there is a large number of, mostly hidden, symmetries which reduce the effective ways a system can evolve. Integrability and solvability are often used as synonymous but actually the precise meaning of each concept needs careful clarification and only after that their connection can be understood. There are many reviews and books at different levels which discuss in deep the meaning of integrability in different context, namely quantum systems and classical ones. Here, we refer to Ref. [15] for integrability from a classical point of view and to Ref. [16] for quantum case. Despite exactly solvable models are rare they neverthless have a special role in our understanding many non-perturbaitive phenomena. Real-life phenomena seem to come out from very complicated and random interactions between many different degrees of freedom. For instance, in a many-body interacting system, from an excess of internal energy in the system, trilions of scattering events between molecules are generated and so chaotic motion. If we let the system evolve subject to its own interactions only, we will see that, after a reasonable amount of time, it will equilibrate to a steady state: we say that the system has reached thermodynamic equilibrium. Traditionally, the approach to many-body complicated systems has seen three different lines of research:

- Approximation schemes
- Perturbation theory
- Numerical Methods

For example, in a typical condensed matter system, electrons interact with each other and with ions on a lattice. The *Hartee-Fock approximation* is a mean field equation which considers each electron as independent, moving in the average potential created by the other electrons. In this approximation ions are considered frozen on the lattice. Instead, if one wants to study vibrational properties of the solid, one can consider oscillations of the ions around their equiilbrium positions. The first approximation is to treat them as classical particles, so harmonic oscillators, and compute variuos natural frequencies. The next step is to quantize the oscillators and study their dispersion relations. This leads to the concept of *phonons*. Following this way electrons may play no role, at least at the beginning, but as we take them into account interesting phenomena may occur: a celebrated example is superconductivity, explained in terms of electrons-phonons interaction [17]. On the other side, perturbation theory has reached a certain level of maturity. Applications of this powerful technique range from particle physics to many-body physics. Despite this, there are a number of drawbacks in using and interpreting perturbation theory. First, there are phenomena that are intrinsecally non-perturbative:

out-of-equilibrium physics and topological physics are two simple examples. Second, mathematically we get power series exapansions which are generically divergent: the best we can do is to interpret them as asymptotic series. Moreover, today we have at our disposal an incredible set of high-performance computational tools. Even smartphones have computational power far higher than an IBM computer had in the seventies. New numerical methods are coming out: the density renormalization group, exact diagonalization, tensor networks and so on [18]. Unfortunately, in many cases, the complexity of macroscopic systems (mostly the quantum ones in $d > 1$) prevents any straightforward application of the aformentioned methods due to entanglement effects (citation). Having said all of that, it is clear that the search and the understanding of integrable models is an important task in modern physics, since it can ultimately give us very useful insights on how interacting many-body systems behave exactly. It is not sufficient, for a true understandig of strong coupling regimes, to exclusively rely on approximate schemes. In addition, a theorem by Liouville [15, 19] ensures that the dynamics of integrable theories is profoundly different with respect to that of non-integrable ones and is reduced to oscillations because they have as much conserved quantities as degrees of freddom. In the thermodynamic limit the situation is similar as the number of conserved charges becomes extensive. Understanding the exact dynamics of integrable theories would provide a comprehension of exact behavior of fully interacting models, without approximations or numerical errors. Put in this way it could seem that the interest in studying integrable models is purely theoretical: this is definitely wrong. On the classical side, integrable field theories arise in many different areas: the Kortweg-deVries partial differential equation describes solitary waves in shallow water, magnetohydrodynamics waves and long waves in anharmonic crystals; the Non-Linear-Schrödinger Equation (NLSE) instead, which, by the way, will be our playground model to test exact predictions, appears to describe homogeneous BEC systems [20, 21] and solitons in fiber optics [6]. In 1955, in Los Alamos Laboratories, Fermi, Pasta and Ulam investigated the approach to equilibrium in an anharmonic chain. The experiment showed that, for initial configurations peaked on low energy sector, quasi-periodic motion was a rule rather than an exception and it seemed to the authors that there was a lack of thermalization, an unexpected situation due to the non-integrability of the system. This can be put in the context of the KAM theorem [22] which allows for quasi-periodic motion in presence of small integrability-breaking terms. However, KAM theorem holds well for systems with finite degrees of freedom and it is not directly applicable in the thermodynamic limit. Later, the situation of the FPU paradox was better explained by Kruskal and Zabusky [23] who recognized that the low energy sector of the FPU model is well captured by the KdV equation. Thus, for very long times integrability of KdV breaks down and non-integrable sector of FPU model kicks in, making the system thermalize. In the quantum world a breakthorugh experiment was performed in 2006, when Kinoshita et al. investigated the equilibration properties of a BEC system varying its dimensionality d [2]. In one dimension this can be depicted as the quantum version of the *Newton's Cradle* as illustrated in Figure 1. They found that the equilibration properties of a one dimensional system of ^{37}Rb bosons is completely different with respect to higher dimensions. Indeed, in $d = 2, 3$ the system thermalized as expected, but in $d = 1$ it did not. The reason was attributed to approximate conservation laws in the model. Indeed, the famous Lieb-Liniger model (LL) [24, 25] nicely fits with the experimental realization of the bosonic system with repulsive contact interactions, despite the presence of the

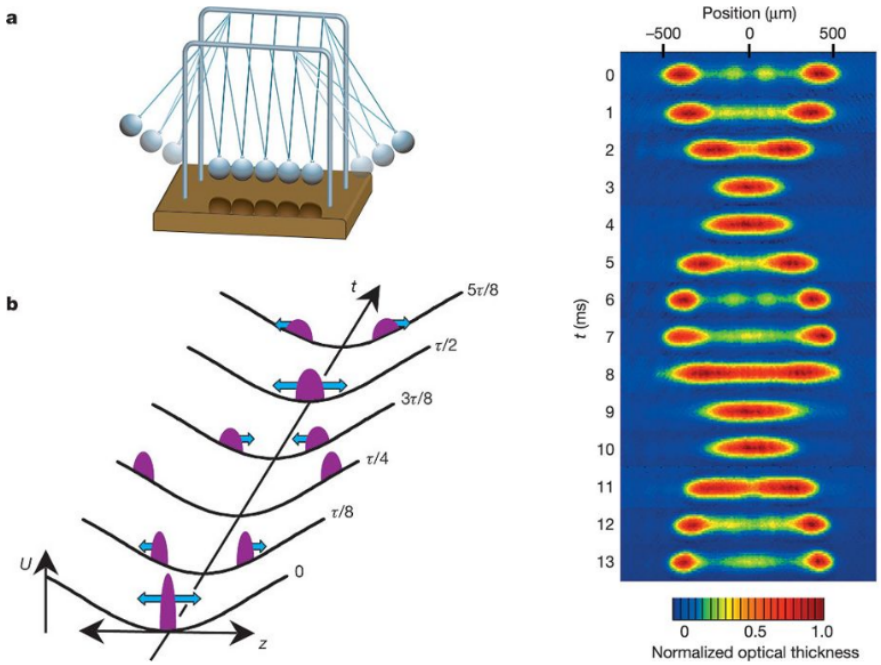

FIGURE 1: (a) Classical Newton's Cradle. (b) Pictorial representation of out-of-equilibrium time evolution of trapped $1d$ BEC considered in [2]. On the right, absorption image of the same time evolution. Figure taken from the same reference.

trapping potential which weakly spoils exact integrability. These kind of experiments are nowdays continuously performed in laboratories, allowing the study of non equilibrium dynamics of nearly isolated quantum systems. The theoretical paradigm through which we explore this phenomenon is the *quantum quench* [3–5], a protocol which has been successfully applied to a lot of many-body systems. To explain the protocol, consider a Hamiltonian $H(g)$ depending on a parameter. At $t < 0$, one prepares the system in the ground state $|\psi_0\rangle$ of $H(g_0)$ with $g \neq g_0$, in such a way that the system is not in an eigenstate of $H(g)$ (and therefore exhibits non trivial evoution), displaying at the same time well defined physical properties like extensivity and the cluster decomposition property. At $t = 0$ the parameter is suddenly changed to another value g and the state evolves according to $H(g)$. Then, one is interested in computing steady state properties of *local* observables [26], namely the $t \to +\infty$ limit. In the Bogolioubov approximation, the NLSE describes the dynamics of a *homogenous* BEC in the ground state [27, 28]. Indeed, handling the semi-classical limit of the LL model amounts to consider the field operator ψ as a classical quantity. Also, in the ground state of the BEC there is a macroscopic number of particles, so that field operators of this state are of order $\sqrt{N}$ and one can ignore their non-commutativity. Thus, studying the out-of-equilibrium properties of the NLSE would provide useful information both on the behavior of the quantum system in such a regime and on the dynamics of the equation itself. The question on the equilibration properties of classical integrable field theories in general has been analyzed in [9] in the context of the Sinh-Gordon model (ShG) [29]. This model is also integrable and contains only one bosonic massive particle, rendering its study relatively simple. In the former reference, the authors considered the $1+1$ dimensional ShG

equation with initial conditions far-from-equilibrium and extensive total energy. This has been achieved enforcing periodic boundary conditions and populating only short wavelength modes of the initial configuration. On one hand, it is well known that quantum integrable models are solvable by means the Thermodynamic Bethe Ansatz [30, 31], which permits the computation of thermodynamic quantities at equilibrium. On the other side, there is a longstanding tradition in mathematical physics community studying classical integrable field equations [32–34] as a complete theory for such models has become well estabilished: this is called Inverse Scattering Method (IST). Subtleties arise with periodic boundary conditions, since the method is supposed to work when fields are rapidly vanishing at infinite distance. The expression for the solution in the thermodynamic limit becomes impossible to use: this is the *infinite gap solution*. An important point of Ref. [9] is that they rised the idea to use the solution of the full quantum problem and a semi-classical limit $\hbar \to 0$ to access quantities in the corresponding classical theory: Form Factors theory [35, 36], TBA and the LeClaire-Mussardo formula valid in [37] and out of equilibrium [38], allow to compute steady-state averages of meaningful physical quantities, like the trace of the stress-energy tensor or the vertex operators. Albeit extremely useful, the LeClaire-Mussardo in an expansion and an exact resummation is extremely difficult. On the other hand, new results in the quantum LL model give access to several physical observables by means of closed and exact expressions. The merit of the above paper is that the authors linked the IST to TBA. Indeed, determining the effective temperatures of the Generalize Gibbs Ensemble (GGE) [39–41]is a tremendous task. However, a huge step forward can be made once one recognizes that an equivalent amount of information is contained in the *transfer matrix* coming from the IST. Once this information has been extracted, numerically for example, it is possible to solve the TBA equations in the semi-classical limit. In Ref. [8, 42] a step forward the computation of one point functions $\langle (\psi^\dagger(x)\psi(x))^K \rangle$ in the LL model has been done: to access the steady state properties they do not use the LeClaire-Mussardo expansion, rather a combination of the recently conjectured Negro-Smirnov formula, which gives the expectation values of vertex operators in the ShG model in arbitrary macrostates [43–45]and can be applied to the quantum LL after a proper non-relativistic limit [46, 47]. At this point it should be clear that we have at our disposal several results both in the quantum setting and in the classical one. Exploiting the knowledge of one point functions in the LL model we can compute the same quantities for the NLSE, after the semi-classical limit. We go further: in [8] the *full counting statistics* (FCS) for the number of particles in a small interval has been computed, but the applicability is limited to a very tiny inteval. The FCS is rather important from the experimental point of view, since it let us compute probabilities of outcomes of a given measure, not just its average value in the classical realm. We calculate exactly the FCS in the steady state through exploiting again the non-relativistic and semi-classical limit of the Negro-Smirnov formula. Our result is valid on equilibirum thermal states and arbitrary GGEs as well, opening the possibility to study the FCS on steady states following arbitrary quenches. Since our model is classical we have the possibility to chek our analitycal calculations using extensive numerical simulations, an impossible task in the quantum setting. The theisis is organized as follows. In Chapter 1 we introduce the concept of integrability in classical mechanics both for systems with finite and infinte degrees of freedom. In particular, we discuss integrable field theories in relation with statistical physics applications and free theories will provide the simplest models displaying non trivial character and most

of the their characteristic will remain in interacting models. Indeed we will se here the Lieb-Liniger model and the coordinate Bethe Ansatz solution leading a quasiparticles description of its Hilber space. Lastly, we introduce the semiclassical limit of quantum fields. In Chapter 2 we explain the problem of relaxation and thermalization in classical integrable field theories. The quench protocol is introduced and applied to the relativistic bosonic free field as an example of non trivial properties of out-of-equilibrium physics of integrable theories. Initial conditions are discussed and it is shown that if the latters are analytic we can make some quantitative prediction about the time scales involved in the process of equilibration. Moreover, the Thermodynamic Bethe Ansatz is deeply reviewed and its semiclassical limit is presented on the grounds of the Lieb-Liniger model: this will serve as a basis for our future calculations. Chapter 3 is devoted to the review of the Inverse Scattering Method, a powerful analytical tool used to find solutions to a class of certain non linear partial differential equations. The method will provide an infinite set of conserved quantities in the the Non Linear Schrödinger model, proving its integrability and, more importantly, will allow us to identify the relevant quantity, the root density, which characterize uniquely the steady state. An important point here, is the relation between the problem defined on the whole line and tha one defined on the circle (period boundary conditions). Chapter 4 is where we compute the full counting statistics for the number of particles and the density one point correlation functions. We present the non relativistic limit of the Sinh-Gordon field theory, resulting in the quantum Lieb-Liniger model. This fact sets up a mapping between the two theories and consequentely a correspondence between their field contents. The full contig statistics is computed by means of a combination of the non relativistic and semiclassical limits of a formula due to Smirnov and Negro allowing the computation of vertex operators in the Sinh-Gordon model. The last chapter is devoted to the numerical recipes we have implemented to compare analytical predictions with direct numerical simulations of the Non Linear Schrödinger equation in out-of-equilibrium conditions.

Chapter 1

Integrable Models

In this Chapter we briefly review the main ingredients of the theory of classical systems with finite degrees of freedom. A formulation of mechanics based on symmetries is natural to show these ideas at work, so the Hamiltonian formulation is the preferred choice. We beign with basic concepts of Hamitlonian mechanics, like canonical tranformations and action-angle variables for simple one dimensional systems. We state and prove Liouville's definition of integrability. We discuss the celebrated KAM theorem and show why it is not directly applicable in the thermodynamic limit. After, we give the idea of why a precise definition of integrability in the quantum setting is still missing. We discuss the role of scattering in integrable models and introduce the concept of S- matrix, as the analytic linear transformation encoding *all* the properties of scattering, especially in $d = 1 + 1$ models. Finally, we introduce a classical integrable models which will play an important role in this thesis: the non-Linear Schrödinger model together with its quantized versions, the Lieb-Liniger model. Morover, we will enlighten an important relation between the famous Sinh-Gordon model [29] and the Lieb-Liniger model. Indeed, the Lieb-Liniger model can be regarded as the non-relativistic limit of the Sinh-Gordon.

1.1 Classical Integrability

1.1.1 Finite Number of Degrees of Freedom

In classical mechanics the mathematical description of a physical system is rather simple. We describe a system with N degrees of freedom by two sets of canonical coordinates $(q_i, p_i) \in \Gamma_N$. We will use bold letters to indicate sets of quantities: for instance, $\mathbf{q}$ stands for all the coordinates q_i. A dot indicates the time derivative. The set Γ_N is a differentiable manifold and is called *phase space*. It is always of even dimension $2N$. With respect to the Lagrangian formualation, the doubling of independent coordinates is a worth price to pay in favour of a theory based on symmetries. For a system with Hamiltonian $H(\mathbf{q}, \mathbf{p})$, the equation of motion take the form,

$$\dot{\mathbf{q}} = \frac{\partial H(\mathbf{q}, \mathbf{p})}{\partial \mathbf{p}} \quad \dot{\mathbf{p}} = -\frac{\partial H(\mathbf{q}, \mathbf{p})}{\partial \mathbf{q}} \tag{1.1}$$

The phase space is promoted to a symplectic manifold with the introduction of the Poisson brackets defined for each pair of observables F and G as,

$$\{F, G\} = \sum_{i=1}^{N} \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right) \tag{1.2}$$

Indeed, we call *observables* every differentiable function $F : \Gamma_N \to \mathbb{R}$ and from the form of equations of motion it soon follows that the time evolution is *generated*[1] by the Hamiltonian,

$$\dot{f} = \{f, H\} \tag{1.3}$$

Equivalently it is possible to define a non-degenerate 2-form $\Omega = \sum_{i=1}^{N} dq_i \wedge dp_i$ acting on vector fields as,

$$\Omega(X, Y) = \{X, Y\} \tag{1.4}$$

Every pair of observables satisfying,

$$\{F, G\} = 1 \tag{1.5}$$

is said to be a canonical pair. Clearly, (q_i, p_i) is a canonical pair for every i. If the Poisson bracket between two obervables vanishes we say that they commute. An important point of this way of writing the evolution equation for an observable is that every time it commutes with the Hamiltonian is a constant of motion. The set of observables equipped with the pointwise sum of functions and the scalar multiplication by a real number becomes a vector space. The introduction of the Poisson bracket is known as Lie Algebra. It satisfies the three basic properties of a Lie product,

1. (Linearity) $\{\alpha F + \beta G, H\} = \alpha\{F, H\} + \beta\{G, H\}$
2. (Anti-simmetry) $\{F, G\} = -\{H, F\}$
3. (Jacobi'd identity) $\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{H, \{F, G\}\} = 0$

A coordinate transformation between a canonical set of coordinate $(\mathbf{q}, \mathbf{p})$ and another canonical set $(\mathbf{Q}, \mathbf{P})$ is said to be canonical. In other words, canonical transformations preserve the hamiltonian structure of the equations of motion. As we said above, the Hamiltonian formulation is more natural to make symmetries explicit. To illustrate this point, without loss of generality, consider a generic observable for a one dimensional system $f = f(q, p)$ and an infinitesimal translation, $q \mapsto q + \epsilon$. It follows that,

$$\delta f = f(q + \epsilon, p) - f(q, p) = \epsilon \frac{\partial f}{\partial q} + O(\epsilon^2) \tag{1.6}$$

Now consider the following Poisson bracket,

$$\{f, \exp(\epsilon p)\} = \{f, 1 + \epsilon p + O(\epsilon^2)\} = \epsilon \frac{\partial f}{\partial q} + O(\epsilon^2) \tag{1.7}$$

By comparison it follows that

$$\delta f = \epsilon\{f, p\} + O(\epsilon^2) \tag{1.8}$$

In this sense the momentum generates infinitesimal translation. Also, infinitesimal translations are canonical. In the very same way the Hamiltonian generates time translations. The argument above in just a particular example of a more general formalism

[1] This is the deep meaning of what is meant by symmetry.

based on what are called *generating functions*. Consider, again for simplicity, a one dimensional Hamiltonian $H(q,p)$, where (q,p) are canonical variables. Equations of motion can be derived from the *Hamilton's principle*

$$\delta \int_{t_1}^{t_2} \left(p\dot{q} - H(q,p)\right) dt = \delta \int_{t_1}^{t_2} \left(P\dot{Q} - K(Q,P)\right) dt = 0 \tag{1.9}$$

which holds when variations at end points are zero. This means that[2],

$$p\dot{q} - H(q,p) = P\dot{Q} - K(Q,P) + \dot{F} \tag{1.10}$$

The function F is called *generating function* of the canonical transofmation and it can depend on any combination of old and new canonical coordinates. For example if $F = F_1(q,Q,t)$ we have,

$$p\dot{q} - H(q,p) = P\dot{Q} - K(Q,P) + \frac{\partial F_2}{\partial q}\dot{q} + \frac{\partial F_2}{\partial p}\dot{p} + \frac{\partial F_2}{\partial t} \tag{1.11}$$

The above relation can hold only if we identify,

$$p = \frac{\partial F_2}{\partial q} \tag{1.12}$$

$$P = -\frac{\partial F_2}{\partial p} \tag{1.13}$$

$$H = K + \frac{\partial F_2}{\partial t} \tag{1.14}$$

If we want to consider $F = F_2(q,P,t)$ the Legendre transform is the answer,

$$F_2(q,P,t) = F_1(q,Q,t) + PQ \tag{1.15}$$

$$p = \frac{\partial F_2}{\partial q} \tag{1.16}$$

$$Q = \frac{\partial F_2}{\partial P} \tag{1.17}$$

Of course there exist other types of generating functions. The Hamilton-Jacobi method of action-angle variables consists in finding a particularly simple set of canonical coordinates: the ones which stay constant in time. For this choice, Hamilton's equations are,

$$0 = \frac{\partial K}{\partial P} = \dot{Q} \tag{1.18}$$

$$0 = -\frac{\partial K}{\partial Q} = \dot{P} \tag{1.19}$$

[2]More generally, $\mu\left(p\dot{q} - H(q,p)\right) = P\dot{Q} - K(Q,P) + \dot{F}$

Hence, the new Hamiltonian must not depend on coordinates at all. If we choose $K = 0$ from (1.14) we get an equation for F,

$$H(q, p, t) + \frac{\partial F}{\partial t} = 0 \tag{1.20}$$

Further, if we take $F = F_2(q, P, t)$ from (1.16) we discover the *Hamilton-Jacobi equation*,

$$H\left(q, \frac{\partial F_2}{\partial q}, t\right) + \frac{\partial F_2}{\partial t} = 0 \tag{1.21}$$

It is customary to rename $F_2 = W$. In the literature this is known as *Hamilton's characteristic function*. Now if the system is time-translation invariant the Hamiltonian is conserved,

$$H(q, p) = \alpha_1 \tag{1.22}$$

and solving for p we find $p = p(q, \alpha_1)$. Thus, the integral,

$$J(\alpha_1) = \oint p dq \tag{1.23}$$

depends only on α_1. This is called action variabale. The integration is extended over an entire period of the motion. Solving for α_1 we can write,

$$H = H(J) \tag{1.24}$$

The Hamiltonian is function only of the action variable. The conjugate variable ω is called angle. The name comes from the fact that this quantity is connected to the freqeuncy of rotation or libration of the system,

$$\omega = \frac{\partial W}{\partial J} \tag{1.25}$$

for the Hamilton's equation are,

$$\dot{\omega} = \frac{\partial H}{\partial J} = v(J) \tag{1.26}$$

This is easily solved,

$$\omega(t) = v(J)t + \beta \tag{1.27}$$

To makes explicit connection with the frequency (or the period) we compute,

$$\Delta\omega = \oint \frac{\partial \omega}{\partial q} dq = \oint \frac{\partial^2 W}{\partial q \partial J} dq = \frac{d}{dJ} \oint \frac{\partial W}{\partial q} dq = \frac{d}{dJ} \oint p dq = 1$$

But we know how ω evolves in time so,

$$\Delta\omega = v(J)T = 1 \tag{1.28}$$

That is,

$$T = \frac{1}{v(J)} \tag{1.29}$$

are the oscillation periods. This can be done if the system is doing periodic motion in phase space, since otherwise we can't define $\oint$-integrals. This is true for an harmonic oscillator, but not for a free particle.

1.1.2 Liouville Integrability and KAM theorem

Having recalled basic concepts of Hamiltonain mechanics for systems with finite degrees of freedoms, we give with the definition of integrability due to Liouville,

Definition 1. *(Liouville integrability) A system with N degrees of freedom is said to be integrable if it has N independent conserved quantities in involution,*

$$\{F_i, F_j\} = 0 \tag{1.30}$$

Independence here means that the set defined by the simultaneous conditions $F_i = f_i \in \mathbb{R}$, $i = 1, \dots, N$ define an N dimensional sub-manifold of the phase space Γ_N^f.

Posed in this way, the definition of integrable system in the classical context is clear. It suffices to find a system of conserved charges in number equal to the degrees of freedom and in involution (in the above sense) to estabilish integrability. The meaning of the existence of such a system of charges is that the dynamics is constrained by the conservation laws. The bijection between conservation laws and degrees of freedom is so stringent that the system can be solved by quadratures. Indeed we have the following theorem [19],

Theorem 1. *(Liouville) The solution of the equations of motion of a Liouville integable system can be obtained by quadratures.*

Solvability means that, instead of solving directly the equations of motion, which in are in general PDEs, we can find a change of coordinates such that the dynamics becomes simple (trivial). By quadrature means that we have to solve a finite number of integrals. The idea behind the theorem is exactly the one described for the Hamilton-Jacobi method of action angle variables. One passes from a set $(\mathbf{q}, \mathbf{p})$ to a new set of canonical coordinates $(\mathbf{F}, \mathbf{\Psi})$ still satisfying,

$$\{F_i, \Psi_j\} = \delta_{ij} \tag{1.31}$$

which trivialize the dynamics according to (1.18)-(1.19). The proof of Liouville's theorem is easy and we report it hereafter [15].

Proof. Define the canonical 1-form as $\alpha = \sum_i p_i dq_i$ and the symplectic 2-form (equivalent to the existence of a Poisson bracket structure on the phase space) as $\Omega = d\alpha = \sum_i dp_i \wedge \mathrm{d}q_i$. The task is to construct a canonical transformation $(q_i, p_i) \to (F_i, \Psi_i)$ such that the F_i are among the new coordinates:

$$\Omega = \sum_i dp_i \wedge dq_i = \sum_i dF_i \wedge \mathrm{d}\Psi_i$$

To construct the transformation we build up its generating function. Let Γ_N^f the level manifold of the phase space $F_i(\mathbf{q}, \mathbf{p}) = f_i$ and suppose that we can solve these relations for p_i. Consider the function,

$$S(\mathbf{F}, \mathbf{q}) \equiv \int_{m_0}^{m} \alpha = \int_{\mathbf{q_0}}^{\mathbf{q}} \sum_i p_i(f, q) dq_i \tag{1.32}$$

where the integration path is drawn from Γ_N^f and goes from the point $m_0 = (\mathbf{p}(\mathbf{f}, \mathbf{q_0}), \mathbf{q_0})$ to the point $m = (\mathbf{p}(\mathbf{f}, \mathbf{q}), \mathbf{q})$ where $\mathbf{q_0}$ is some reference value. Assume the existence of S. It follows that we can define,

$$\Psi_i = \frac{\partial S}{\partial F_i}$$

and we have,

$$dS = \sum_i \Psi_i dF_i + p_i dq_i$$

Since $d^2 S = 0$ we find $\Omega = \sum_i dp_i \wedge dq_i = \sum_i dF_i \wedge d\Psi_i$. This proves that S is well-defined and the transformation canonical. To prove that such an S exists we must show that it is not dependent on the integration path. By Stokes theorem, we need,

$$d\alpha|_{\Gamma_N^f} = \Omega|_{\Gamma_N^f} = 0$$

To do this one consider the vector field associated to F_i defined by $dF_i = \Omega(X_i, \circ)$,

$$X_i = \sum_j \frac{\partial F_i}{\partial q_j}\frac{\partial}{\partial p_j} - \frac{\partial F_i}{\partial p_j}\frac{\partial}{\partial q_j}$$

These vector fields are tangent to the level manifold Γ_N^f because of Liouville integrability,

$$X_i(F_j) = \{F_i, F_j\} = 0$$

Since F_i are assumed to be independent this tangent space is well defined. Also $\Omega(X_i, X_j) = dF_i(X_j) = 0$ and we have proved that $\Omega|_{\Gamma_N^f} = 0$. □

Having recalled and proved the Liouville's theorem we now discuss the KAM theorem in a simplified manner, in order to avoid technical detours which exile the purposes of this thesis. This classical theorem concerns with the stability of motions in hamiltonian systesm that are small perturbations of integrable ones. They are generically described by Hamiltonians of the type,

$$H = H_I + \epsilon V \tag{1.33}$$

where H_I is an integrable Hamiltonian while ϵ is the strenght of the perturbation. Before KAM theorem, it was believed that resonant tori in the phase space are destroyed

by arbitrairily small perturbations. If H_I is integrable it depdens only by the action variable defined in the previous section, $H_I = H_I(J)$, with $J \in D \subset \mathbb{R}^N$. Unperturbed Hamilton's equation are readily integrated to give,

$$J = J_0 \quad \omega(t) = \omega_0 + v(J)t$$

with $\partial_J W = \omega \in [0, 2\pi]$ is the angle variable and S is the Hamilton's characteristic function. The geometry of the flow of an integrable system is thus described as periodic cycles on an n-dimensional torus $\mathbb{T}^N$ with periods given by (1.29). One should not forget that the original coordinates are related to the action-angle ones by a certain transformation, but the motion is anyway periodic in the angle variables. Thus, it is possible to expand the solutions in Fourier series and they will be of the form,

$$\sum_{k \in \mathbb{Z}^N} a_k(J) e^{i(\langle k, \omega_0 \rangle + t\langle k, v(J) \rangle)} \tag{1.34}$$

As a consequence, the motion is *quasi-periodic* in t ($\langle , \rangle$ denotes the usual scalar product and we are meaning vector quantities). The frequences of the motion can be of two types [48],

- Non-resonant or rationally independent $\langle k, v(J) \rangle \neq 0$ for all $k \neq 0 \in \mathbb{Z}^N$
- Resonant or rationally dependent $\langle k, v(J) \rangle \neq 0$ for some $k \neq 0 \in \mathbb{Z}^N$

When one considers the perturbed Hamiltonian H, it can be proven that the majority of tori survives. Not only their frequencies are resonant but *strongly* resonant, meaning that there exist constants $\alpha > 0$ and $\tau > 0$ such that,

$$|\langle k, v \rangle| \geq \frac{\alpha}{|k|^\tau} \quad \forall 0 \neq k \in \mathbb{Z}^N \tag{1.35}$$

with $|k| = \sum_i |k_i|$ where i ranges over the number of frequences (for a multi-frequency motion). This condition is called *diophantine* or *small divisor* condition. The KAM theorem states the stability of invariant tori provided that,

$$|\epsilon| < \delta \alpha^2 \tag{1.36}$$

for some $\delta > 0$. It is clear that, since k's are integers, as we take the thermodynamic limit the above bound vanishes and the theorem loses its validity. For what concerns the dynamics of field theories we cannot realy on any precise mathematical theory as the KAM theoy. Studies are based uniquely on numerical methods and for what concerns integrable models on the Inverse Scattering Transform.

1.2 Integrable Field Theories

I the Fifties it has been recognized that a theory of fundamental interactions is best built on Lagrangians supporting infinite degrees of freedom. This led to the development of quantum electrodynamics and later, with the introduction of local gauge invariance, models for the weak and strong interactions. Before that, there were at least two important physical models based on the field theoretic paradigm, namely the theory of

General Relativity and Maxwell's Electrodynamics. Soon, the field theoretic point of view was extendend to the description of condensed matter systesms, an incredible arena of unexplained phenomena to test this languange. Like in classical or standard quantum mechanics, each model is specified giving a Lagrangian or a Hamiltonian, this time depending on an infinite number of degrees of freedom. The main guiding line for the construction of such models has been the symmetry principle: in short, each system is characterized by a particular set of symmetries and one builds up the Lagrangian (or better the Action) having the same symmetries. Thanks to Noether's Theorem, for each symmetry there are associated conserved charges. These charges can be usually derived from an integral over a local density, which in turn is the first component of a current satisfying a continuity equation. Our focus is on a particular class of field theories which we call *integrable*. At the very primitive level we will see that these theories display an infinite set of conserved charges.

1.2.1 Prototype of Integrable Field Theory: free models

Non Relativistic Free Field

As one of the simplest field theories one can conceive, we present the free theory of a single complex bosonic field $\psi(x,t)$ in $1+1$ dimensions ruled by the Hamiltnoinan,

$$H = \int \mathcal{H}(\psi,\bar{\psi})\mathrm{d}x = \int |\partial_x \psi(x,t)|^2 \mathrm{d}x \tag{1.37}$$

and Lagrangian,

$$L = \int \mathcal{L}(\psi,\bar{\psi})\mathrm{d}x = \int \left\{ \frac{i}{2}\left(\bar{\psi}\partial_t\psi(x,t) - \psi\partial_t\bar{\psi}(x,t)\right) - |\partial_x\psi(x,t)|^2 \right\} \mathrm{d}x \tag{1.38}$$

The integration is over the real line for fields vanishing at infinity or over a period for periodic fields. The Poisson structure is given by the following bracket,

$$\{\psi(x,t),\bar{\psi}(y,t)\} = i\delta(x-y) \tag{1.39}$$

with the vanishing of the remaining brackets. This is because the canonical momentum associated to the variable ψ is,

$$\pi(x,t) = \frac{\partial\mathcal{L}}{\partial(\partial_t\psi)} = i\bar{\psi}(x,t) \tag{1.40}$$

The field equation corresponding to this theory is the free-particle Schrödinger equation,

$$i\partial_t\psi(x,t) = -\partial_x^2\psi(x,t) \tag{1.41}$$

For the moment we do not specify boundary conditions for this equation. From now, it is understood tha the field depends always on t as we will display only the spatial coordinate for notational simplicity. Despite this theory may seem very simple since there is no interaction, it neverthless displays many of the concepts that we will find in non trivial interacting models. In is easy to show that in this theory there are at least

two conserved quantities in addition to the Hamiltonian itself,

$$P = -i \int \bar{\psi}(x)\partial_x \psi(x) \mathrm{d}x \tag{1.42}$$

$$N = \int |\psi(x)|^2 \mathrm{d}x \tag{1.43}$$

which are interpreted as the total momentum and number of particles respectively. The first follows from invariance under translations while the second is a consequence of $U(1)$ symmtetry. Now, introduce the Fourier Transform as,

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int e^{ikx} A(k) \mathrm{d}k \tag{1.44}$$

These amplitudes satisfy,

$$\{A(q), \bar{A}(k)\} = i\delta(k-q) \tag{1.45}$$

and again other brackets vanish. Introducing in the Hamiltonian we find,

$$H = \frac{1}{2\pi} \int \epsilon(k) |A(k)|^2 \mathrm{d}k \tag{1.46}$$

where $\epsilon(k) = k^2$ is the non relativistic dispersion relation. Note that we have diagonalized the Hamiltonian in action-angle variables: the action is given by $J(k) = |A(k)|^2$ and the angle by $\epsilon(k)$. We find also,

$$P = \int p(k) |A(k)|^2 \mathrm{d}k \tag{1.47}$$

$$N = \int |A(k)|^2 \mathrm{d}k \tag{1.48}$$

with $p(k) = k$. The reason for writing conserved quantities in this way will become clear in a moment. Define the quantities,

$$\mathcal{Q}_n = (-i)^{n-1} \int \bar{\psi}(x) \partial_x^{n-1} \psi(x) \mathrm{d}x \tag{1.49}$$

Then, for every $n \in \mathbb{N}$ these are *local* conserved quantities in the free theory,

$$\{H, \mathcal{Q}_n\} = 0 \quad \forall n \in \mathbb{N} \tag{1.50}$$

The proof is straightforward and is based on standard rules of functional calculus and on the fact that boundary terms after integration by parts vanish due to boundary conditions,

$$\frac{\delta H}{\delta \bar{\psi}(x)} = -\partial_x^2 \psi(x) \tag{1.51}$$

$$\frac{\delta H}{\delta \psi(x)} = -\frac{\delta}{\delta \psi(x)} \int \bar{\psi}(x) \partial_x^2 \psi(x) \mathrm{d}x \tag{1.52}$$

$$= -\int \partial_x^2 \bar{\psi}(x,t) \delta(x-x) \mathrm{d}x \tag{1.53}$$

$$= -\partial_x^2 \bar{\psi}(x) \tag{1.54}$$

Also,

$$\frac{\delta \mathcal{Q}_n}{\delta \bar{\psi}(x)} = (-i)^{n-1} \partial_x^{n-1} \psi(x) \tag{1.55}$$

$$\frac{\delta \mathcal{Q}_n}{\delta \psi(x)} = i^{n-1} \partial_x^{n-1} \bar{\psi}(x) \tag{1.56}$$

So that,

$$\begin{aligned} \{H, \mathcal{Q}_n\} &= i \int \left[-(-i)^{n-1} \partial_x^2 \bar{\psi}(x) \partial_x^{n-1} \psi(x) + i^{n-1} \partial_x^2 \psi(x) \partial_x^{n-1} \bar{\psi}(x) \right] \mathrm{d}x \\ &= i \int \left[-(i)^{n-1} \bar{\psi}(x) \partial_x^{n+1} \psi(x) + (-i)^{n-1} \bar{\psi}(x) \partial_x^{n+1} \psi(x) \right] \mathrm{d}x = 0 \end{aligned} \tag{1.57}$$

Using the Fourier Transform to express the quantities we find that,

$$\mathcal{Q}_n = \int q_n(k) J(k) \mathrm{d}k \tag{1.58}$$

where $q_n(k) = k^n$ is called charge *eigenvalue*. From equations of motion we can easily see that,

$$A(k) = A_0(k) e^{i\epsilon(k)t} \tag{1.59}$$

From this simple expression it is explicitely seen that the value of the conserved charges is fixed once the initial field configuration is given, since the value $A_0(k)$ will be given too. In this way expression (1.58) is suggestive since it gives conserved charges as a sum of independet contributions. This will be important in a moment when we will discuss thermodynamics of integrable models. Since now we have keep the discussion completely classical. In the quantum setting we assume the same hamiltonian and the same lagrangian. Calssical fields are promoted to quantum operators. Here the $O^\dagger$ indicates the hermitian conjugate of an operator O. Canonical quantization is accomplished by the rule,

$$\{\circ, \circ\} \mapsto \frac{1}{i\hbar}[\circ, \circ] \tag{1.60}$$

The canonical conjugate of ϕ this time is,

$$\pi(x,y) = i\psi^\dagger(x,t) \tag{1.61}$$

Thus, we obtain,

$$[\psi(x,t), \psi^\dagger(y,t)] = \delta(x-y) \tag{1.62}$$

and all other commutator vanish exactly as before. The field is expanded in Fourier components as,

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int a(k) e^{ikx} dk \tag{1.63}$$

with,

$$[a(k), a^\dagger(q)] = \delta(k-q) \tag{1.64}$$

These operators are the creation and annihilation operators of quantum field theory. From the expression of the energy we find that In particular, expression (1.58) becomes,

$$\mathcal{Q}_n = \int q_n(k) a^\dagger(k) a(k) \tag{1.65}$$

Notice that the charges are *local* also in the quantum setting. The analogy with the harmonic oscillator of quantum mechanics defines the Hilbert space of the theory. In particular, the vacuum state is that annihilated by all the a's,

$$a(k)\,|0\rangle = 0 \tag{1.66}$$

In this way the vacuum carries zero value of all the conserved charges, not only the energy,

$$\mathcal{Q}_n\,|0\rangle = 0 \tag{1.67}$$

Particle states are constructed acting with creation operators on the vacuum. For example the one particle state is,

$$a^\dagger(k)\,|0\rangle = |k\rangle \tag{1.68}$$

A convenient notation used to represent multiparticle states is that of the occupation numbers. If there are n_k particles in the state k we write $|\ldots n_k \ldots\rangle$ so that a multiparticle state with N particles is,

$$|n_{k_1}, \ldots, n_{k_m}\rangle = \prod_{j=1}^{m} \frac{\left(a^\dagger(k_j)\right)^{n_{k_j}}}{\sqrt{n_{k_j}!}}\,|0\rangle \tag{1.69}$$

with $\sum_{j=1}^{m} n_{k_j} = N$. The normalization factor takes into account the Bose statistics of the particles. Importantly,

$$n(k_j) = a^\dagger(k_j) a(k_j) \tag{1.70}$$

is the operator which counts the number of particles in state k_j analogous to $|A(k)|^2$ in the classical case.

Relativistic Free Field

The relativistic hermitian bosonic field of mass m in $1+1$ dimensions is ruled by the Klein-Gordon lagrangian density,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right) \tag{1.71}$$

where $a_\mu b^\mu = a_0 b^0 - a_0 b^1 = a_t b^t - a_x b^x$ is the usual lorentz invariant product. The discussion of the relativistic field is very similar to the previous case the difference being consequences of lorentz invariance. This time we go straight working directly with the quantum formalism, since it should be clear that most of the manipulations we do are valid in the classical theory too. The canonical momentum is,

$$\pi(x,t) = \partial_t \phi(x,t) \tag{1.72}$$

The canonical commutator is,

$$[\phi(x,t), \pi(y,t)] = i\delta(x-y) \tag{1.73}$$

The hamiltonian density reads,

$$\mathcal{H} = \frac{1}{2}\left(\pi^2 + (\partial_x \phi)^2 + m^2 \phi^2\right) \tag{1.74}$$

Field equations are given by,

$$(\partial_t^2 - \partial_x^2 - m^2)\phi = 0 \tag{1.75}$$

Also in this case they are solved by Fourier transform. The only difference is that the dispersion relation is $\epsilon(k) = \sqrt{k^2 + m^2}$. Since the field is self-conjugate ($\phi = \phi^\dagger$) we can write[3],

$$\phi(x,t) = \frac{1}{\sqrt{2\pi}} \int \frac{\mathrm{d}k}{\sqrt{2\epsilon(k)}} \left\{ a(k) e^{ikx - i\epsilon(k)t} + a^\dagger(k) e^{-ikx + i\epsilon(k)t} \right\} \tag{1.76}$$

with,

$$[a(k), a^\dagger(q)] = \delta(k-q) \tag{1.77}$$

The annihilation operator kills the vacuum,

$$a(k)\,|0\rangle = 0 \tag{1.78}$$

and the multiparticle Hilbert space is constructed exactly as in (1.69). The hamiltonian is readily diagonalized,

$$H = \frac{1}{2}\int \mathrm{d}k \epsilon(k)[a^\dagger(k)a(k) + a(k)a^\dagger(k)] = \int \mathrm{d}k \epsilon(k) a^\dagger(k) a(k) + \frac{1}{2}\int \mathrm{d}k \epsilon(k)\delta(0) \tag{1.79}$$

Here comes the QFT problem of UV infinities. The delta function at zero is a bad divergence. To solve the problem we suppose that the theory is ill-defined at the beginning and introduce the normal ordering of fields: put every annihilation operator on the

[3]The lorentz invariant integration measure is, $\frac{\mathrm{d}k}{\sqrt{2\epsilon(k)}}$

right and every creation operator on the left. In general to normal order an operator one does $: O := O - \langle 0|O|0\rangle$, so that expectation values on the ground state of normal ordered operators is zero. Also, the vacuum energy is zero. This is justified saying that we cannot measure energies but only differences. Also note that the delta function once regularized on the lattice becomes proportional to $1/\Delta$ where Δ is the lattice spacing that is why it is a UV i.e. short distance divergence. Our normal ordered hamiltonian is,

$$: H := \int \mathrm{d}k \epsilon(k) a^\dagger(k) a(k) \tag{1.80}$$

To see that also this model displays infinite conserved charges we introduce the light cone variables,

$$\sigma = x - t, \quad \tau = x + t \tag{1.81}$$

In these variables the equation of motion is,

$$\partial_\sigma \partial_\tau \phi = \frac{1}{4} m^2 \phi \tag{1.82}$$

Form here we can construct an infinite sequence of continuity equations,

$$\partial_\tau [\partial_\sigma^n \phi]^2 = \frac{1}{4} m^2 \partial_\sigma [\partial_\sigma^{n-1} \phi]^2, \quad \partial_\sigma [\partial_\tau^n \phi]^2 = \frac{1}{4} m^2 \partial_\tau [\partial_\tau^{n-1} \phi]^2 \tag{1.83}$$

which have the form $\partial_\tau A = \partial_\sigma B$. Going back to variables (t, x) they become of the type $\partial_t (A + B) = \partial_x (A - B)$. This leas to the conserved quantities $\int \mathrm{d}x (A + B)$,

$$\mathcal{Q}_n = \int \mathrm{d}x \left\{ (\partial_\sigma^n \phi)^2 + \frac{1}{4} m^2 (\partial_\sigma^{n-1} \phi)^2 \right\} \tag{1.84}$$

$$\mathcal{Q}_{-n} = \int \mathrm{d}x \left\{ (\partial_\tau^n \phi)^2 + \frac{1}{4} m^2 (\partial_\tau^{n-1} \phi)^2 \right\} \tag{1.85}$$

From these we can construct even and odd charges as,

$$\mathcal{E}_n = \mathcal{Q}_n + \mathcal{Q}_{-n} \tag{1.86}$$

$$\mathcal{O}_n = \mathcal{Q}_n - \mathcal{Q}_{-n} \tag{1.87}$$

Note that,

$$\mathcal{E}_1 = H = \frac{1}{2} \int \mathrm{d}x \left\{ \pi^2 + (\partial_x \phi)^2 + m^2 \phi^2 \right\} \tag{1.88}$$

All these charges have UV divergences analogous to the hamiltonian. To make sense of their action on states we need the normal ordering to avoid ambiguities. Adding and subtracting the expressions and defining the rapidity variable as $\epsilon(k) = m \cosh(\theta)$, $k = m \sinh(\theta)$, we find,

$$: \mathcal{E}_n := : \mathcal{Q}_n + \mathcal{Q}_{-n} := \frac{m^{2n-1}}{2^{2n-1}} \int \mathrm{d}\theta J(\theta) \cosh[(2n-1)\theta) \tag{1.89}$$

$$: \mathcal{O}_n := : \mathcal{Q}_n - \mathcal{Q}_{-n} := \frac{m^{2n-1}}{2^{2n-1}} \int \mathrm{d}\theta J(\theta) \sinh[(2n-1)\theta) \tag{1.90}$$

where we have defined the action variable,

$$J(k) = ma^{\dagger}(m\sinh\theta)a(m\sinh\theta)\cosh\theta \tag{1.91}$$

The angle variable is again represented by the dispersion relation $\epsilon(k)$. Irrespective of being in the quantum or classical setting, what we learn from these very basic facts are the following,

1. Free theories can be diagionalized in terms of action-angle variables.
2. Free theories display an infinite set of local conservation laws.
3. The Hilbert space is a collection of independent particles, each one identified by its momentum. Each particle is an excitation above the ground state $|0\rangle$.

In the classical case there are, of course, no Hilbert space and no particles, but we can think of modes $A(k)$ mirroring the quantum mechanical situation, that is $A(k)$ can be interpreted as an "annihilation" field. In discussing the statistical mechanics of classical field theories we will see that this is indeed a deep and fruitful connection. The three points above are rather important as we will see that integrable fully interacting theories display the same characteristics albeit with appropriate caveats.

1.2.2 Thermodynamics of Fields

In classical mechanics, for a system described by N generalized coordinates and N generalized momenta, the phase space is a finite $2N$-dimensional manifold Γ_N. In a closed and isolated system one introduces a probability measure on the manifold for states of the system with energy E,

$$p(E) = \frac{1}{\Omega(E)} \tag{1.92}$$

$$\Omega(E) = \int \delta(E - H(q,p))d^N q d^N p \tag{1.93}$$

This probability measure is called *microcanonical ensemble*. The integration is over all the phase space. It is nothing more than the volume occupied by the system in the phase space. Due to the complicated geometry of the domain of integration, the computation of the above integral is in many cases prohibitive for finite N. Approximation and asymptotic tools are often employed to extract the large N dependence of probabilities, since one is interested in this limit. A central quantity in statistical mechanics is the entropy. There exist many definitions of entropy [49], each with its own benefits. The Boltzmann entropy, defined in the context the the microcanonical ensemble is,

$$S(E) = \log\Omega(E) \tag{1.94}$$

The entropy is assumed to be extensive with the system size and by definition is a strictly increasing convex function of E. This has an important consequence on its interpretation. Indees, $\exp S(E)$ is the number of states at energy E. The larger the entropy the more probable is the state. This is the *principle of entropy maximization*. ESM has achieved a certain level of rigor [50], that is why our attention in this thesis turns to

non-equilibrium statistical mechanics. When putting the system in contact with an external heat bath at fixed temperature one defines the *canonical ensemble*,

$$p_\beta^{GE}(x) = \frac{1}{\mathcal{Z}(\beta)} \exp\left(-\beta H(x)\right) \tag{1.95}$$

$$\mathcal{Z}^{GE}(\beta) = \sum_x p_\beta^{GE}(H(x)) \tag{1.96}$$

with the index x labelling the microscopic states of the system. The factor $\mathcal{Z}^{GE}(\beta)$ is called *partition function* and from it one can derive any thermodynamic quantity at equilibrium. The inverse temperature β is fixed by energy conservation, that is, from,

$$E(t=0) = \langle H \rangle_\beta = \sum_x p_\beta^{GE}(x) H(x) \tag{1.97}$$

The above distribution can be derived from the principle of entropy maximization with the constraint that average energy is fixed. Any other statistical ensemble, with some fixed average value, is derived in this way: maximize the entropy functional $S[\rho] = -\int \rho \log(\rho)$ subject to constraints of the form $\sum_j \mu_j \langle \mathcal{F}_j \rangle$, where $\mathcal{F}_j$ is some observable and ρ the stationary to be sought. When j varies in a set of fixed cardinality we call these ensembles (aka statistical distributions over the system) *Gibbs ensembles* or *thermal states*. For field theories the situation is quite the same. Suppose a theory is described by a field ϕ and its conjugate momentum π. Thus, the thermal partition function reads,

$$\mathcal{Z}^{GE}(\beta) = \int \mathcal{D}(\phi, \pi) e^{-\beta H(\phi,\pi)} \tag{1.98}$$

and the thermal average of an observable,

$$\langle O \rangle_\beta = \frac{1}{\mathcal{Z}^{GE}(\beta)} \int \mathcal{D}(\phi, \pi) e^{-\beta H(\phi,\pi)} O(\phi, \pi) \tag{1.99}$$

Integrable models like the ShG and the NLSE have a peculiar property already stressed: they are integrable. We have discussed some of the consequences this status bears. They are special in many respects and we want to deepen the relationship between integrability and convergence towards equilibrium. Gibbs Ensembles are defined at equilibrium, that is they are time independent distributions. Anyway, given an initial condition the system evoleves in time according to some evolution. What ESM assumes is that at time $t = \infty$ the knowledge of the initial condition is lost and the average properties of the system are described by some Gibbs Ensemble. The definition of a probability distribution on the phase space is equivalent to a probability distribution on the initial data: indeed a theorem by Liouville, again, states that the Hamiltonian evolution of the system preserves volumes on the phase space. Thus, the initial volume occupied by the system (a given set of initial conditions) will occupy the same space at later times. What happens for integrable models is quite different and has been recognized for quantum systems first [51]. The ideas is to push forward the entropy maximization principle: this time there will be an infinite set of conserved quantities

and lagrange multipliers. Then, one consider the generalized Hamiltonian defined as,

$$H(x) = \sum_i \mu_i \mathcal{Q}_i(x) \tag{1.100}$$

where $\mathcal{Q}_i$ are the local conserved quantities. The steady-state distribution is called *Generalized Gibbs Ensemble* and reads

$$p_{\underline{\mu}}^{GGE}(x) = \frac{1}{\mathcal{Z}^{GGE}(\underline{\mu})} \exp\left(-H(x)\right) \tag{1.101}$$

where again x is the state of the system. Also, μ_i are fixed by the values of conserved quantities at the initial time,

$$\langle \mathcal{Q}_i \rangle_{\underline{\mu}} = \mathcal{Q}_i(t=0) \tag{1.102}$$

Note, by the way, the the above conditions leads to an infinite set of simultaneous equations to be solved to find the values of the Lagrange multipliers μ_i. The fragmentation of the phase space due to these contraints implies immediately ergodicity breaking, preventig us from using Gibbs ensembles to predict infinite-time averages. It should be remarked that this implementation of GGE is too naive in many instances and has been subject to extensive theoretical investigation [52]. Again, the generalization to field theories is natural and the average of an observable reads,

$$\langle O \rangle_{\underline{\mu}} = \frac{1}{\mathcal{Z}^{GGE}(\underline{\mu})} \int \mathcal{D}(\phi, \pi) e^{-H(\phi,\pi)} O(\phi, \pi) \tag{1.103}$$

where this time H is the generalized Hamiltonian involving the weighted sum of the infinite conserved quantities, this time functionals of the canonical variablesϕ and π. For sake of completeness, we describe also statistical ensembles in quantum mechanics. Indeed, there is a correspondence between quantum mechanics in $d+1$ dimensions and classical statistical mechanics in d dimensions. This correspondence cannot be overestimated, as it can provide an exact mapping between known solutions in the two descriptions. In quantum mechanics we describe the state of a system with a state vector $|\psi(0)\rangle$ which, in the Schrödinger picture, evolves according to the Hamiltonian H,

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle \tag{1.104}$$

This vector is in general a linear combination of other states in the Hilbert space spanned by a complete set states. In the Heisemberg picture, the time evoultion is transferred to observables,

$$O(t) = e^{iHt} O(0) e^{-iHt} \tag{1.105}$$

Anyway, the expectation value is independent on the representation and i si given by,

$$\langle O \rangle(t) = \langle \psi(t) | O(0) | \psi(t) \rangle = \langle \psi(0) | O(t) | \psi(0) \rangle \tag{1.106}$$

Such states are called *pure*, giving rise to genuine quantum effects. If the state of the system is not known with certainty, we define the *density matrix* defined as,

$$\rho = \sum_n p_n |\psi_n\rangle \langle \psi_n|, \quad \sum_n p_n = 1, \quad 0 \leq p_n \leq 1 \tag{1.107}$$

which introduces further classical statistical fluctuations on top of the quantum ones. The density matrix defines the state of the system and satisfies $\rho^\dagger = \rho$ and $\mathrm{Tr}\,\rho = 1$, where Tr is the trace. Also, it can be proven that ρ is a pure state if and only if $\rho^2 = \rho$. The time evolution of this operator is,

$$\rho(t) = e^{-iHt}\rho(0)e^{iHt} \tag{1.108}$$

Expectation values are computed as,

$$\langle O\rangle(t) = \mathrm{Tr}\,\rho(t)O \tag{1.109}$$

The quantum thermal partition function is defined as,

$$\mathcal{Z}(\beta) = \mathrm{Tr}\,e^{-\beta H} = \sum_n \langle\psi_n|e^{-\beta E_n}|\psi_n\rangle \tag{1.110}$$

where $|\psi_n\rangle$ is a complete set of Hamiltonian eigenstates. From this the thermal density matrix is constructed taking $p_n = \mathcal{Z}^{-1}e^{-\beta E_n}$. Following standard approaches [17], the quantum partition function can be represented as a path integral over fields as[4],

$$\mathcal{Z}(\beta) = \int_{\psi(x,0)=\psi(x,\beta)} \mathcal{D}\psi e^{-\int_0^\beta d\tau \int_0^L dx \mathcal{L}_E(\psi,\partial\psi)} \tag{1.111}$$

This formula can be obtained by the following considerations. The quantum amplitude between two field configurations at different times $t_a < t_b$ is,

$$\langle\psi_b, t_b\,|\,\psi_a, t_a\rangle = \langle\psi_b|\,e^{-iH(t_b-t_a)}\,|\psi_a\rangle = \int \mathcal{D}(\psi)e^{i\int \mathrm{d}t\mathrm{d}x\mathcal{L}} \tag{1.112}$$

where $\mathcal{L}$ is the lagrangian density. while the quantum partition function is given by,

$$\mathcal{Z}(\beta) = \mathrm{Tr}\left(e^{-\beta H}\right) = \sum_a \langle\psi_a|\,e^{-\beta H}\,|\psi_a\rangle \tag{1.113}$$

If we analytically continue,

$$(t_b - t_a) = -i\beta \tag{1.114}$$

and trace over $\psi_a = \psi_b$ we find exactly the partition function. The mnemonic rule to get the euclidean lagrangian is the following,

$$\mathcal{L} = \mathcal{T} - \mathcal{V} \mapsto \mathcal{T} + \mathcal{V} = \mathcal{L}_E \tag{1.115}$$

Indeed if we define the *euclidean time* τ by $t = -i\tau$ we have $\partial_t = \frac{\partial\tau}{\partial t}\partial_\tau = i\partial_\tau$ and we get for the kinetic term of a relativistic lagrangian,

$$\mathcal{T}(t) \propto \partial_\mu\psi\partial^\mu\psi = (\partial_t\psi)^2 - (\partial_x\psi)^2 = -(\partial_\tau\psi)^2 - (\partial_x\psi)^2 \propto -\mathcal{T}(\tau) \tag{1.116}$$

[4]Here we use a bosonic path integral. If the field is a fermion we woulg get anti-periodiciy of the field in the imaginary time τ: $\psi(x,0) = -\psi(x,\beta)$.

and factorizing the minus sign we get the exponential factor

$$\exp\left(-\int_0^\beta d\tau \int_0^L dx \mathcal{L}_E\right) \tag{1.117}$$

For a non-relativistic Lagrangian the kinetic term becomes,

$$-\mathcal{T}_E \propto -\left(\psi^\dagger(x)\partial_t\psi(x) - \psi(x)\partial_t\psi^\dagger(x)\right) - |\partial_x\psi(x)|^2 \tag{1.118}$$

Averages on GGEs states is defined in the same way, the only difference it that we use the generalized Hamiltonian (1.100) to define the partition function. It is clear that in the quantum case the situation becomes even harder with respect to the classical one. Later on, we will see how we can get the (1.99) handling a semi-classical limit of (1.113).

1.2.3 Thermal Averages and Free Energy

Now that we have introduced the basic formalism of thermal ensembles and GGEs we want to work out some easy example. Our reference models will be again the free theories. In what follows, fields will be completely classical quantities. The non relativistic hamiltonian with one bosonic field of mass $m = 1/2$ at chemical potential μ,

$$H = \int_0^L \mathrm{d}x \left\{|\partial_x\psi|^2 - \mu|\psi|^2\right\} \tag{1.119}$$

The partition thermal partition function is,

$$\mathcal{Z} = \int \mathcal{D}(\psi,\bar{\psi}) e^{-\beta\int_0^L \mathrm{d}x\left\{|\partial_x\psi|^2 - \mu|\psi|^2\right\}} \tag{1.120}$$

The partition function is easily computed discretizing the theory on a lattice with N points so that $L = Na$ and $\psi(ja) = \psi_j$, where a is the lattice spacing. The discretized Hamiltonian reads,

$$H = a\sum_{j=0}^{N-1}\left\{\frac{|\psi_{j+1} - \psi_j|^2}{a^2} - \mu|\psi_j|^2\right\} \tag{1.121}$$

Introducing the Fourier Transform as,

$$\psi_j = \frac{1}{\sqrt{N}}\sum_{s=0}^{N-1} e^{i\frac{2\pi}{N}js} A_s \tag{1.122}$$

we find,

$$H = a\sum_{s=0}^{N-1}\epsilon_s A_s \bar{A}_s \tag{1.123}$$

with,

$$\epsilon_s = \frac{2}{a^2}\left(1 - \cos\left(\frac{2\pi}{N}s\right)\right) - \mu \tag{1.124}$$

This is the dispersion relation on the lattice. It is immediate to compute,

$$\langle \bar{A}_s A_q \rangle_\beta = \frac{\delta_{q,s}}{a} \frac{1}{\beta \epsilon_s} \tag{1.125}$$

In the continuum limit we have,

$$\langle \bar{A}(s) A(q) \rangle_\beta = \delta(q-s) \frac{1}{\beta \epsilon(s)} \tag{1.126}$$

where, as usual, $\epsilon(k)$ it the non relativistic dispersion relation,

$$\epsilon(k) = k^2 - \mu \tag{1.127}$$

arising, this time, from a power expansion up to second order in N^{-1} of the dispersion on the lattice. Thus,

$$E = \langle H \rangle_\beta = a \sum_{s=0}^{N-1} \epsilon_s \langle \bar{A}_s A_s \rangle_\beta = \frac{1}{\beta} \sum_{s=0}^{N-1} 1 = \frac{N}{\beta} \tag{1.128}$$

We have found that on thermal states, in the limit $N \to +\infty$, the energy is divergent. As a by-product we obtain that the temperature is the energy per degree of freedom,

$$\lim_{N \to +\infty} \frac{E}{N} = \frac{1}{\beta} \tag{1.129}$$

We can go beyond. Indeed, from subsection 1.2.1 we know the expression of conserved charges. On thermal states we find,

$$\langle \mathcal{Q}_n \rangle_\beta = \int \mathrm{d}k q_n(k) \langle |A(k)|^2 \rangle_\beta = \frac{1}{\beta} \int \mathrm{d}k q_n(k) \frac{1}{\epsilon(k)} \tag{1.130}$$

The integrand for large momenta behaves as k^{n-2}. This means that expecation values of local conserved charges on thermal states are UV divergent. Here a comment is important. Let us look at the free energy,

$$\beta \mathcal{F}(\beta) = \log(\mathcal{Z}) = \mathcal{F}_0 + L \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}k}{2\pi} \log(\beta \epsilon(k)) \tag{1.131}$$

where we have used the periodicity of (1.124) and absorbed irrelevant constants in $\mathcal{F}_0$. In the continuum limit, $a \to 0$ we find,

$$\beta \mathcal{F}(\beta) = \mathcal{F}_0 + L \left(a^{-1} \log(\beta a^{-2}) + \log\left(\pi^2\right) \right) + O(a^2) \tag{1.132}$$

From this expression it is seen that the free energy is divergent in the continuum limit. From these simple computations we learn that in free theories the energy and the free energy are meaningful quantities when computed on thermal states because they have meaning only when we put the theory on the lattice. Consider now the relativistic free

field with hamiltonian,

$$H = \int \mathrm{d}x \left\{ \frac{1}{2}(\pi^2 + (\partial_x \phi)^2 + m^2\phi^2) \right\} \tag{1.133}$$

By virtue of what has been showed in subsection 1.2.1 the main ingredients remain unaltered due to the fact that the expressions for the local conserved charges are formally the same as the non relativistic case. They are obtained by weighting the charge eigenvalue $q_n(k)$ with the function $n(k) = \langle |A(k)|^2 \rangle$. In the quantum setting this would have been the *occupation number function* which counts the average number of particles in the state k. In the integrability literature this is also called *filling fraction* and we will stick to this tradition. The free energy of the relativistic system has the same form too. The only difference is the degree of divergence with the lattice spacing, which, for example, in the case of the free energy this time is worst and goes like a^{-1}. The thermodynamics of free model is rather simple and can be solved rather easily. Indeed computing thermal averages of arbitrary product of fields is not a hard task since, being the distributions of the fields gaussian, we can invoke Wick's theorem. Expectation values of charges are characterized by the filling fraction, which, in the classical setting plays the role as in quantum mechanics. Now that we have seen basic features of classical models we want to disccuss in brief the integrability quantum scenario as where will be an entire chapter dedicated to classical integrable field theories. We will see that the Hilbert space in quantum integrable models can be thought as a collection of *quasiparticles* in exactly the same way we have done in free theories. Actually, as we have already said, there is much more and we will go into in in a moment.

1.2.4 The Lieb-Liniger model

We introduce here, for completeness and future usefulness, the Lieb-Liniger model (LL) which describes a one dimensional quantum system of non-relativistic bosons interacting via a delta function potential, deferring a deep discussion of its classical version, the Non Linear Schrödinger model, to Chapter 3. In the realm of quantum cold atoms this is a very special model. It is used to fit data from many experimental realistic situations [2]. At the theoretical level, the LL model is integrable and as such has an infinite number of conservation laws [53] and the ground state and excitation spectrum at zero temperature were fully characterized by Lieb and Liniger [24, 25] with the so-called Bethe Ansatz. The thermodynamics of the model at finite temprature was described by Yang and Yang via a generalization of the Bethe Ansatz, today referred as Thermodynamic Bethe Ansatz (TBA) [30]. The Hamiltonian of the LL model reads,

$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \partial_{x_i}^2 + 2g \sum_{i<j} \delta(x_i - x_j) \tag{1.134}$$

The second quantized version is easily obtained in terms of field operators as,

$$H = \int_0^L dx \left[-\frac{\hbar^2}{2m} \psi^\dagger(x) \partial_x^2 \psi(x) + g \psi^\dagger(x) \psi^\dagger(x) \psi(x) \psi(x) \right] \tag{1.135}$$

with,

$$[\psi(x), \psi^\dagger(y)] = \delta(x-y) \quad [\psi(x), \psi(y)] = [\psi^\dagger(x), \psi^\dagger(y)] = 0 \tag{1.136}$$

Here, note that the imaginary unit is absent, due to the operatorial nature of the field. The quantization rule is exactly the same of that of ordinary quantum mechanics. Classical fields are promoted to quantum operators and the Poisson bracket is mapped into the commutator according to $\{\circ, \circ\} \mapsto \frac{[\circ,\circ]}{i\hbar}$. At first sight, the Hamiltonian admits at least two conserved quantities, the number of particles and the total momentum,

$$N = \int_0^L \psi^\dagger(x)\psi(x)dx$$
$$P = -i\hbar \int_0^L dx \psi^\dagger(x)\partial_x\psi(x)$$

Actually there is an infinite set of conserved quantities, see [54]. We focus on the repulsive regime $g > 0$, since the attractive case is physically pathologic because the energy is not bounded from below. It should be stressed that the effective coupling constant is given by,

$$\lambda = \frac{2mg}{\hbar^2 n} \tag{1.137}$$

where g is the coupling appearing in the Hamiltonian while $n = N/L$ is the density of particles. The definition of the effective coupling follows immediately from dimensional analysis: it is intensive and dimensionless and as such a good candidate to provide informations about the actual strenght of the interaction. The $\lambda = +\infty$ is called Tonks-Girardeau model and describe a system of impenetrable bosons while for $\lambda = 0$ we recover free bosons. The effective coupling can be experimentally tuned by Fleshbach resonances or controlling the density [55]. The LL model was introduced by Lieb and Liniger with the hope to have a fully interacting quantum many body system exactly solvable. This was relevant in many respects: first, it could be used to test approximate expressions coming from perturbation theory; second, the known Tonks-Girardeau model was a zero parameter model in the sense that the spectrum was not significantly sensible to any change or parameters appearing in the model (the density and the radius of bosons). Here, the effective coupling renders the spectrum sensible on it and more interesting effects can be produced. The field equation arising from the LL Hamiltonian is precisely the NLSE. There exists a generalization of this equation, called Gross-Pitaevski equation, describing the dynamics of the order parameter of a *three dimensional* bosonic system in presence of an external potential [21]. The three dimensional version of the quantum Hamiltonian (1.134) can be replaced with the classical one which in turn describes a homogeneous BEC system in particular conditions:

1. Low enough temperature
2. The number of particles in the condensate N should be large enough
3. The scattering length is small compared to the density $a \ll n^{1/3}$

Neverthless, we restrict our attentiontion to the one dimensional case, so that no condensation actually occurs. The situation is analogous to what happens in electromagnetism. When there is a large number of photons Maxwell's equation become a good

description. Interestingly here there is a difference to be mentioned. While in electromagnetism there is no dependence of the Planck's constant, in this case it appaers explicitely in field equations. In turn, this means that coherence phenomena like interference depend on it [21].

1.2.5 Coordinate Bethe Ansatz

The Bethe Ansatz (BA) technique owes its name to the nobel prize German physicist Hans Bethe. In 1931, he deviced the method while working on a one dimensional spin chain, the antiferromagnetic Heisenberg model. He computed exactly the spectrum and the eigenvectors and since then BA was applied to many one dimensional Hamiltonians. It turns out that the method has deep connections and analogies with the Inverse Scattering Method as we will see. To start, we review in detail the application of the Bethe Ansatz to the LL model. Despite many textbooks on the subject, like [31], the crystal clearer exposition of the solution of LL the model can be found in the original papers [24, 25]. The one dimensional bosonic system described by the Hamiltonian (1.134) was studied in detail by means of BA by Lieb and Liniger. After that, it became known under the name LL model. To see how simple the idea behind the Bethe Ansatz is, we will first analyze the two-particle problem. For simplicity we set $\hbar = 2m = 1$.

2-body problem

The Hamiltonian is,

$$H = -\partial_{x_1}^2 - \partial_{x_2}^2 + 2g\delta(x_1 - x_2) \tag{1.138}$$

Since we have a system of bosons, the wave function $\psi(x_1, x_2)$ must be symmetric under particle permutations. In general we can write,

$$\Psi(x_1, x_2) = f(x_1, x_2)\theta(x_1 - x_2) + f(x_2, x_1)\theta(x_2 - x_1) \tag{1.139}$$

The "Bethe Ansatz" is,

$$f(x_1, x_2) = A(k_1, k_2)e^{i(k_1x_1 + k_2x_2)} + A(k_2, k_1)e^{i(k_2x_1 + k_1x_2)} \tag{1.140}$$

This form is suggested by the fact that in the two sectors of particle orderings, $x_1 < x_2$ and $x_2 < x_1$, the solution is a plane wave. The numbers k_i are called *quasi-momenta* and this function resembles the one of the free particle problem. Indeed, they cannot be true wave vectors, since the momentum is the generator of space translations. The next step is to apply the Hamiltonian to the Ansatz wave function. With the abbreviations $\partial_i = \partial_{x_i}$, $f(x_i - x_j) = f_{ij}$,

$$\begin{aligned}
&-\partial_1^2 \left(f_{12}\theta_{12}\right)) \\
&\quad = -\partial_1^2 f_{12}\theta_{12} - f_{12}\partial_1^2\theta_{12} \\
&\quad = k_1^2 f_{12}\theta_{12} + \delta_{12}\partial_1 f_{12} \\
&\quad = k_1^2 f_{12}\theta_{12} + i\delta_{12}\left(k_1 A_{12}e^{i(k_1x_1 + k_2x_2)} + k_2 A_{21}e^{i(k_2x_1 + k_1x_2)}\right)
\end{aligned} \tag{1.141}$$

In the same way,

$$\begin{aligned}&-\partial_2^2\left(f_{12}\theta_{12}\right))\\&\quad=k_2^2 f_{12}\theta_{12}-i\delta_{12}\left(k_2 A_{12}e^{i(k_1x_1+k_2x_2)}+k_1A_{21}e^{i(k_2x_1+k_1x_2)}\right)\end{aligned}\tag{1.142}$$

The symmetry $1\leftrightarrow 2$ immediately gives the same results for $-\partial_i^2(f_{21}\theta_{21})$. Thus,

$$H\Psi=(k_1^2+k_2^2)\Psi+2\delta_{12}\left[g(A_{12}+A_{21})+i(A_{12}-A_{21})(k_1-k_2)\right]e^{2i(k_1+k_2)x_1}\tag{1.143}$$

where we have used the presence of the delta function to set $x_1=x_2$ in (1.141)-(1.142). In order for the wave function to be an eigenstate it must be,

$$\frac{A_{12}}{A_{21}}=\frac{i(k_1-k_2)-g}{i(k_1-k_2)+g}\tag{1.144}$$

Using the defintion,

$$\arctan(z)=\frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)=\frac{1}{2i}\log\left(\frac{i-z}{i+z}\right)$$

we can write,

$$\frac{A_{12}}{A_{21}}=\frac{i(k_1-k_2)-g}{i(k_1-k_2)+g}=\frac{i\frac{k_1-k_2}{g}-1}{i\frac{k_1-k_2}{g}+1}=\frac{i+\frac{k_1-k_2}{g}}{-i+\frac{k_1-k_2}{g}}=-\frac{i+\frac{k_1-k_2}{g}}{i-\frac{k_1-k_2}{g}}$$

and finally,

$$\begin{aligned}\frac{A_{12}}{A_{21}}&=\exp\left[\log\left(-\frac{i+\frac{k_1-k_2}{g}}{i-\frac{k_1-k_2}{g}}\right)\right]\\&=\exp\left\{i\pi+\log\left[\left(\frac{\frac{k_1-k_2}{g}}{i-\frac{k_1-k_2}{g}}\right)^{-1}\right]\right\}\\&=\exp(i(\theta(k_1-k_2)+i\pi)=\exp(i\tilde{\theta}(k_1-k_2))\end{aligned}\tag{1.145}$$

$$\theta(k)=-2\arctan\left(\frac{k}{g}\right)\tag{1.146}$$

$$\tilde{\theta}(k)=\theta(k)+\pi\tag{1.147}$$

Above, we have chosen the branch-cut of the complex logarithm so that $\log(-1)=\pi$. The definition of the *scattering phase* (1.146) as an odd function of its argument is standard in the literature on the subject. From this result we read out two important things: the first is that probability is conserved since $|A_{12}|=|A_{21}|$; the second is that the S-matrix, $S(k)=\exp(i\tilde{\theta})$ is a pure phase, depends only on the difference between the two momenta and $S(0)=-1$. The last comment is rather important since it can be recognized that $S(0)$ is the scattering matrix of free fermions. These simple results generalize to scattering between more particles and they are a direct manifestation of

integrability: scattering is factorized in 2-body processes and particle production (i.e. 3-body processes) does not occur.

N-body problem

Now we solve the general problem with N particles. The Bethe Ansatz wavefunction is,

$$\Psi\left(\{x\};\mathcal{Q}\right)=\sum_{\sigma\in\mathcal{S}_N}A(\sigma;\mathcal{Q})\exp\left(i\sum_{j=1}^{N}k_{\sigma j}x_j\right) \tag{1.148}$$

Here $\mathcal{S}_N$ is the symmetric group of N elements and $\mathcal{Q}$ indicates a particular particle ordering. In the notations we have used to solve the 2-body problem,

$$\Psi\left(\{x\};x_1<x_2\right)=f(x_1,x_2)$$

Note that the fact the the system is one dimensional is crucial: in two dimensions particles cannot be ordered since on $\mathbb{R}^2$ it is not possible to define a *total order*. The bosonic symmetry of the wave function implies that we can work at fixed particle order: exchanging two coordinates does nothing. Further, a different ordering can be reached just applying a permutation to two different momenta at time. Thus, we consider the sector $x_i<x_j$ for $i<j$ and apply the Hamiltonian (1.134) to the wave function (1.148),

$$H\Psi(\{x\})=\left(\sum_{j=1}^{N}k_j^2\right)\Psi(\{x\}) \tag{1.149}$$

Also, notice that the total momentum is easily given by,

$$P\Psi(\{x\})=-i\sum_{i=1}^{N}\partial_i\Psi(\{x\})=\left(\sum_{j=1}^{N}k_j\right)\Psi(\{x\}) \tag{1.150}$$

To understand the wave function structure in different ordering sectors, it suffices to consider how amplitudes associated to different permutations of momenta are related. Reminiscent again of the 2-body problem for $\sigma,\sigma'\in\mathcal{S}_N$ exchanging k,k' and leaving all the others k's unchanged we have,

$$\frac{A(\sigma)}{A(\sigma')}=e^{i\tilde{\theta}(k-k')}=-e^{i\theta(k-k')} \tag{1.151}$$

where $\tilde{\theta}$ is defined in (1.147) and θ in (1.146). This relation is obtained by appling the Hamilotinian to (1.148), isolating terms containing k and k' and using the delta function constraint. Iteration starting from the identity permutation I (for which $A(I)=1$) gives,

$$A(\sigma)=C\epsilon_\sigma\exp\left(i\sum_{j<l}\theta(k_{\sigma j}-k_{\sigma l})\right) \tag{1.152}$$

where ϵ_σ is the sign of the permutation and C a normalization constant. Indeed, if σ is the permutation mapping $\{k\}$ into $\{k'\}=\{k'_{\alpha_1},\ldots,k'_{\alpha_N}\}$, rearranging $\{k'\}$ into $\{k\}$ by

transposing only adjacent k's, we get a factor $-\exp(i\theta(k'_s - k'_t)$ for each transposition (k'_s is to be on the left of k'_t before the transposition). The contact interaction gives just the form of the wave function but does not determine the quasi-momenta. To quantize the system we do as usual: we put it in a box of length L and impose periodic boundary conditions,

$$\Psi(x_1, \ldots, x_j + L, \ldots, x_N) = \Psi(x_1, \ldots, x_j, \ldots, x_N) \quad j = 1, \ldots, N \tag{1.153}$$

Equivalenty, we can ask that the wave function is the same at points $x_j = 0$ and $x_j = L$. This is true for the full wave function. We can write for example,

$$\Psi(0, x_2, \ldots, x_N) = \Psi(L, x_2, \ldots, x_N) \tag{1.154}$$

But $L > x_2$ so that this conditons can't be used directly on wave function restricted to the domain we are considering. From Bose symmetry we know that,

$$\Psi(L, x_2, \ldots, x_N) = \Psi(x_2, \ldots, x_N, L) \tag{1.155}$$

Now the arguments of the right hand side are in the right order. This point is rather subtle and often omitted in many textbooks and reviews on Bethe Ansatz. With the help of the last observation we can write the boundary conditions for the wave function restricted to $x_1 < \cdots < x_N$ as,

$$\Psi(0, x_2, \ldots, x_N) = \Psi(x_2, \ldots, x_N, L) \tag{1.156}$$

The requirement must hold for any x_j indepentently. Setting $x_1 = 0$ kills every x_1-dependent exponential in the wave function leaving out only amplitudes (or scattering phases that is equivalent). Since exponentials are linearly independet functions we must have a relation between amplitudes. Setting $x_N = L$ let survive all the exponential terms like,

$$e^{iLk_{\sigma N}}$$

Finally, (1.156) must be valide for any x_s so we find,

$$\exp\left(ik_{\sigma s}L\right) = (-1)^{N-1} \exp\left(i \sum_{l=1}^{N} \theta(k_{\sigma s} - k_l)\right) \tag{1.157}$$

Since permutations are bijections, $\sigma s = j$, leading to an equation valid for any $j \in \{1, \ldots, N\}$,

$$\exp\left(ik_j L\right) = (-1)^{N-1} \exp\left(i \sum_{l=1}^{N} \theta(k_j - k_l)\right) \tag{1.158}$$

This condition can be physically understood by the following observation: moving a particle around a circle of length L will make it acquire a kinematic phase $\tilde{\theta}$ associated to each scattering event with all the other particles, plus a dynamical pahse kL because it has walked the entire circle, see Fig. 1.1. Taking the logarithm gives the celebrated *Bethe equations*:

$$k_j L = 2\pi \tilde{I}_j + (N-1)\pi + \sum_{i=1}^{N} \theta(k_j - k_i) \tag{1.159}$$

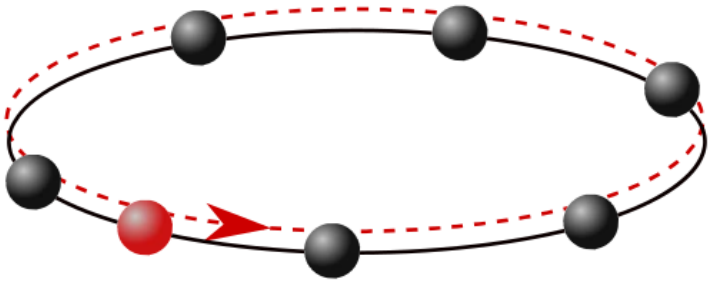

FIGURE 1.1: Physical picture behind (1.160).

Given a set of integers $\tilde{I}_j$, for every $j \in \{1,\dots,N\}$, k_j is determined solving this $N \times N$ algebraic system. We will use both *roots* and quasi-momenta to refer to solutions of Bethe equations. For practical purposes it is better to define a new set of interegers $I_j = \tilde{I}_j + \frac{N-1}{2}$ such that Bethe Equations can be written as,

$$k_j L = 2\pi I_j + \sum_{i=1}^{N} \theta(k_j - k_i) \tag{1.160}$$

The numbers I_j are called *Bethe quantum numbers* and uniquely specify the state of the system. They are are integers or half-integers if N is odd or even respectively. In the limit $g \to +\infty$ we have free fermions and all the I_j have to be different. For free particles $k_i = \frac{2\pi I_i}{L}$ have to be all different too. Since the scattering phase is continuous in g^{-1}, by (1.160), also the k_i's are continuous functions of it: we conclude that, also when $g \neq 0$, for different I_i's there are different k_i's. It is easy to demonstrate that an ordering on the I_j's implies an ordering on the k's. Indeed, subtracting two Bethe equations gives,

$$L\delta_{sj} = L(k_s - k_j) = 2\pi(I_s - I_j) + \sum_{l=1}^{N} \left[\theta(k_s - k_l) - \theta(k_j - k_l)\right] \tag{1.161}$$

Now, we order the I's and consider $I_s > I_j$. Being the scattering phase monotonically decreasing, the sum on the right hand side of the last equation would be positive leading to $\delta_{sj} > 0$: it must be also $\delta_{sj} > 0$. The converse statement is proven in the same way. Introducing the "action",

$$\mathcal{S}(k_1,\dots,k_N) = \frac{1}{2}L\sum_{i=1}^{N} k_i^2 + 2\pi\sum_{i=1}^{N} k_i I_i - \frac{1}{2}\sum_{i,j=1}^{N} \theta(k_i - k_j) \tag{1.162}$$

and minimizing it, we see that the solution is also unique. Moreover, by the same continuity argument as above the minimum energy is obtained when the integers are symmetrically distributed around 0,

$$I_j = -\frac{N+1}{2} + j \quad j \in \{1,\dots,N\} \tag{1.163}$$

since this is true for free fermions. This clearly defines a state of total zero momentum,

$$P = \sum_{i=1}^{N} k_j = \sum_{i=1}^{N} \frac{2\pi I_i}{L} + \sum_{i=1}^{N}\sum_{l=1}^{N} \frac{\theta(k_i - k_l)}{L} = \sum_{i=1}^{N} \frac{2\pi I_i}{L} = 0 \tag{1.164}$$

because $\theta(-k) = -\theta(k)$. From this rather brief discussion of the coordinate Bethe Ansatz we should learn the basic lesson. Even if the model is interacting its spectrum can be exactly found. This is ultimately due to the infinitely many local charges: bethe states are common eigenvectors of all of them. Pictorially, Bethe eigenstates can be imagined as particles on the circle scattering in a non trivial way: when particle i scatters particle j the associated amplitude gets multiplied by a phase $S(k_i - k_j) = \exp(i\tilde{\theta}(k_i - k_j))$. After an entire circle the quantization condition is,

$$\prod_{j=1}^{N} e^{i\tilde{\theta}(k_k - k_j)} = e^{ik_k L} \tag{1.165}$$

The factor $S(k)$ is the S-matrix, which here is a simple scalar function. Scattering is completely factorized in two body processes, a typical occurence in quantum integrable models[5]. Quasi-momenta are quantized and in one to one correspondence with integers numbers. In turn, eigenfunctions are superpositions of plane waves with non trivial amplitudes in the region $x_1 < \cdots < x_N$. Every state in the Hilbert space can be written as,

$$|\lambda_1, \ldots, \lambda_N\rangle = \frac{1}{\sqrt{N!}} \int \mathrm{d}^N x \Psi_N(x_1, \ldots, x_N | \lambda_1, \ldots, \lambda_N) \psi^\dagger(x_1) \ldots \psi^\dagger(x_N) |0\rangle \tag{1.166}$$

where Ψ is the Bethe eigenfunction (1.148) and λ's satisfy the Bethe equations. Moreover, as the interaction strenght is set to zero we recover usual momenta for free particles,

$$k_n = \frac{2\pi}{L} n \tag{1.167}$$

All of this has the flavour of a free-like theory. For instance, we will see later, when we will take the thermodynamic limit, that epressions for conserved quantities are analogous to that of free hamiltonians considered before. The thermodynamics of the model at finite temperature will let us begin our journey on exploring the steady state properties of integrable theories.

A comment on the scattering phase

Let us comment the scattering phase of the Bethe Ansatz solution. In fact, as noted in [30], extracting the logarithm in (1.158) is somewhat subtle, having the possibility to arbitrarily choose the branch of the logarithm. We already have chosen a particular branch in (1.147). It is well known that polydromy of the logarithm (and of the most part of complex functions) can be traced out in the ambiguity in the definition of the function $Arg(z)$, the argument. If we choose $Arg(z)$ such that taking the logairthm in (1.145)

[5] See also Appendix A.2.

compensates for the π appearing in (1.146) we get $\tilde{\theta} \equiv \theta$. In this case, $S(0) = 1$: the S-matrix of free bosons. This has implications in the possible values of Bethe quantum numbers. Indeed if the system is bosonic, the fermionic scattering phase is in contrast with the Bose statistics and as a consequence quantum numbers have to be all different. Conversely, the bosonic scattering phase allows for many integers to be the same. We will proceed with the fermionic fomulation of the Bethe Ansatz. It is possible to work with both formulations and each one comes with its pros and cons. We will return on this on later sections when we will face the problem of performing the semi-classical limit of the Bethe equations.

1.3 Semiclassical limit of Quantum Fields

In this section we will see that the semiclassical limit of the quantum partition function leads to the classical one. To get non trivial and finite expressions we have to extract the leading order in $\hbar$ of different quantities involved. Taking the limit $\hbar \to 0$ is actually a mathematical trick to exctract the semiclassical behavior. Indeed, physical constants of Nature are constant and it is not physical to set $\hbar \to 0$. As can be seen in eq. (1.185) our limit is equivalent to a high temperature and small coupling limit. We do this explicitely on the Lieb-Liniger model, arriving to the Non Linear Schrödinger hamiltonian.

1.3.1 Semiclassical Limit in Statistical Field Theory

Recall the hamiltonian and the lagrangian density of the LL-model (keeping the mass and $\hbar$ explicit),

$$\mathcal{H} = \frac{\hbar^2}{2m}|\partial_x \psi(x)|^2 + g\psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x) \tag{1.168}$$

$$\mathcal{L} = \frac{i\hbar}{2}\left(\psi^\dagger(x)\partial_t\psi(x) - \psi(x)\partial_t\psi^\dagger(x)\right) - \frac{\hbar^2}{2m}|\partial_x\psi(x)|^2 - g\psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x) \tag{1.169}$$

The partition function of a classical field theory,

$$Z^{cl}(\beta) = \int \mathcal{D}\psi\mathcal{D}\pi e^{-\beta H(\psi,\pi)} \tag{1.170}$$

and of course there is no appearance oh $\hbar$. In the quantum setting, we have seen that, if $\hbar = 1$, for a bosonic theory with a single field $\psi = \psi(x,\tau)$, we have,

$$Z^q(\beta) = \int_{\psi(x,0)=\psi(x,\beta)} \mathcal{D}\psi e^{-\int_0^\beta d\tau \int_0^L dx \mathcal{L}_E(\psi,\partial\psi)} \tag{1.171}$$

where $\mathcal{L}_E$ is the euclidean Lagrangian obtained from the lagrangian changing the sign of the kinetic energy. In order to get the classical partition function from the quantum one we need to perform the semiclassical limit. To do this non trivially it is better to write explicitely the definition of the quantum path integral measure,

$$\mathcal{D}\psi = \prod_{ij} d\psi_i(\tau_j) \tag{1.172}$$

where i is the spatial index and $\tau_j = \frac{\beta}{N} j$ with $j \in \{0, \dots, N\}$ is the euclidean time discretization. For the moment it is not necessary to specify the range of the spatial index. Periodic boundary conditions on the path integral $\psi(x,0) = \psi(x,\beta)$ translates into $\psi_i(\tau_0) = \psi_i(\tau_N)$ for every i. In the limit $\hbar \to 0$, after rescaling the Planck's constant $\beta \to \hbar\beta$, the exponential factor (1.117) at leading order becomes

$$\exp\left(-\hbar\beta \int_0^L dx \mathcal{L}_E(\psi(x,0), \partial\psi(x,0))\right) \tag{1.173}$$

How does the path integral measure change in this limit? We have,

$$\mathcal{D}\psi = \prod_i d\psi_i(\tau_0) \dots d\psi_i(\tau_N) \approx \prod_i d\psi_i(0) d\psi_i(\hbar\beta) \tag{1.174}$$

We make the change of variables $\psi_i(\hbar\beta) \mapsto \psi_i(\hbar\beta) - \psi_i(0) \approx \hbar\beta\partial_\tau\psi_i(\tau)|_{\tau=0}$ and we find, apart constant factors which do no contribute to expectation values,

$$\mathcal{D}\psi \overset{\hbar\to 0}{\approx} \prod_i d\psi_i(0) d\partial_\tau\psi_i(\tau)|_{\tau=0} \equiv \mathcal{D}\psi\mathcal{D}\partial_\tau\psi \tag{1.175}$$

For relativistic theories usually $\partial_\tau\psi \propto \pi$ while for non-relativistic ones[6] $\mathcal{D}\partial_\tau\psi$-integration factors out in expectation values of local observables, so that in both cases we have established the correct way the measure is transformed into the classical integration over the phase space. Indeed, the term linear in time derivatives of non-relativistic theories in (1.173) gives,

$$\psi^\dagger(0)\partial_\tau\psi(\tau)|_{\tau=0} \tag{1.176}$$

In the semi-classical limit, we are discretizing the whole interval $[0, \hbar\beta]$ with only two points, namely $\tau = 0$ and $\tau = \hbar\beta/2$, and since we have the constraint of periodic boundary conditions, at zero order in $\hbar$ we can approximate, $\partial_\tau\psi(\tau)|_{\tau=0} \approx \frac{\psi(\hbar\beta)-\psi(0)}{\hbar\beta} = 0$. Thus, this term does not contribute to the euclidean action. Put in another way, we can discretize the τ integral,

$$\begin{aligned}\int_0^{\hbar\beta} \psi^\dagger(\tau)\partial_\tau\psi(\tau)d\tau &\approx \frac{\hbar\beta}{2}\left[\psi^\dagger(0)\left(\frac{\psi(\hbar\beta/2)-\psi(0)}{\hbar\beta/2}\right) + \psi^\dagger(\hbar\beta/2)\left(\frac{\psi(\hbar\beta)-\psi(\hbar\beta/2)}{\hbar\beta/2}\right)\right] \\ &= -\left(\psi(\hbar\beta/2)-\psi(0)\right)\left(\psi^\dagger(\hbar\beta/2)-\psi^\dagger(0)\right) \\ &\approx \left(\frac{\hbar\beta}{2}\right)^2 \partial_\tau\psi\partial_\tau\psi^\dagger \end{aligned} \tag{1.177}$$

From this we see that the term in (1.176) is subleading in $\hbar$ compared to the purely spatial term of the lagrangian, which in this limit gets simply multiplied by $\hbar\beta$.

[6] Non-relativistic field theories, in order to admit wave solutions, must have at least a complex scalar field.

Actually, this rather heuristic argument can be precisely justified going in Matsubara frequency space. Introducing the Fourier Transform for the bosonic field,

$$\psi(x,\tau) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{+\infty} \psi_n(x) e^{-i\omega_n \tau} \tag{1.178}$$

$$\psi_n(x) \equiv \psi(x,\omega_n) = \frac{1}{\sqrt{\beta}} \int_0^\beta e^{i\omega_n \tau} \psi(x,\tau) d\tau \tag{1.179}$$

with $\omega_n = \frac{2\pi n}{\beta}$, the euclidean action,

$$\begin{aligned}
\mathcal{S}_E &= \int_0^\beta d\tau \int_0^L dx \mathcal{L}_E \\
&= \int_0^\beta d\tau \int_0^L dx \Big\{ \frac{1}{2} \Big(\psi^\dagger(x,\tau) \partial_\tau \psi(x,\tau) - \psi(x,\tau) \partial_\tau \psi^\dagger(x,\tau) \Big) \\
&\quad + \frac{1}{2m} |\partial_x \psi(x,\tau)|^2 + g \psi^\dagger(x) \psi^\dagger(x,\tau) \psi(x) \psi(x,\tau) \Big\} \\
&= \sum_{n=-\infty}^{+\infty} \int_0^L dx \Big\{ - i\omega_n |\psi_n(x)|^2 \\
&\quad + \frac{1}{2m} |\partial_x \psi_n(x)|^2 + \frac{g}{\beta} \sum_{k,l,m=-\infty}^{+\infty} \psi_n^\dagger(x) \psi_k^\dagger(x) \psi_l(x) \psi_m(x) \delta_{n+k+l+m,0} \Big\}
\end{aligned} \tag{1.180}$$

Now it is clear that if $\beta \mapsto \hbar\beta$ also $\omega_n \mapsto \omega_n/\hbar$ and taking $\hbar \to 0$ the imaginary term in the euclidean action will cause rapid oscillations in the path integral exponential factor. As a consequence the frequency $\omega_n = 0$ dominates and we see explicitely that the linear term in time derivatives disappears. This implies also that the path integration over $\mathcal{D}\partial_\tau \psi$ factors out since the action does not depend on time derivatives anymore. Now, consider the quantum thermal expectation values of a generic observable $O(\psi(x))$,

$$\langle O(\psi(x)) \rangle_q^\beta = \frac{1}{Z_q(\beta)} \int \mathcal{D}\psi O(\psi(x)) e^{-\mathcal{S}_E} \tag{1.181}$$

and the same expectation but for $O(\hbar^{1/2}\psi)$,

$$\langle O(\hbar^{1/2}\psi) \rangle_q^{\hbar\beta} \approx \frac{1}{Z_q(\beta)} \int \mathcal{D}\psi O(\hbar^{1/2}\psi) e^{-\hbar\beta \int_0^L dx \mathcal{H}} \tag{1.182}$$

rescaling $g \mapsto \hbar g$ and changing variables in the path integral to $\psi \mapsto \hbar^{-1/2}\psi$ we find,

$$\mathcal{H} \mapsto \frac{1}{\hbar} \mathcal{H} \tag{1.183}$$

Thus, we finally find,

$$\langle O(\hbar^{1/2}\psi(x)) \rangle_q^{\hbar\beta} \approx \frac{1}{Z_{cl}(\beta)} \int \mathcal{D}\psi \mathcal{D}\pi O(\psi(x)) e^{-\beta \int_0^L dx \mathcal{H}(\psi,\pi)} \tag{1.184}$$

This is true because since the field is complex $\mathcal{D}\psi \equiv \mathcal{D}\psi \mathcal{D}\psi^\dagger$ and $\psi^\dagger \propto \pi$. We have

performed the semiclassical limit of the quantum Lieb-Liniger model at the the partiton function level. The important point is that the correct scalings to perform the limit non trivially are,

$$\beta \mapsto \hbar\beta \quad g \mapsto \hbar g \quad O(\psi) \mapsto O(\sqrt{\hbar}\psi) \tag{1.185}$$

These scalings are rather important and should be kepts in mind because they will form the basis for the derivation of basic equations to study out-of-equilibrium dynamics of classical fields.

Chapter 2

Relaxation, Equilibration and Thermalization

As stated many times, we are interested in the study of classical, $(1+1)$ dimensional, non-relativistic, integrable field theories, with particular attention to that represented by the NLSE. Theories may be classified according to their symmetries, to the dimensionality of space-time and for being continuous of discrete. In lattice theories continuous symmetries are usually broken by the lattice. If $d > 1+1$ integrability is broken [56]. The former is a very special feature of the model under consideration and it is often difficult to infer. The study of relaxation properties of classical field theories were initiated in the work by Fermi, Pasta and Ulam [57]. Their findings stimulated a great interest in the theoretical and mathematical physics community: the problem was that a discrete system of anharmonic oscillators seemed to violate energy equipartition at late times. They analyzed the energy spectrum of the chain when the initial configuration was peaked on the slowest mode and noted that energy was shared only among few slow modes (long wave lengths). Also, they observed a periodic exchange of energy between these few modes. This was a big surprise: since the system was believed to be ergodic, due to the hanarmonicity, equipartition was expected. Later, Zabusky and Kruskal [23] showed that the low energy sector of the FPU problem is well approximated by the KdV equation. Indeed, in the paper the appearence of (super-)recurrence times is explained in terms of solitonic interactions. Basically a soliton is wave-like solution of a non-linear equation with extremely nice scattering properties. When two solitons interact they exchange their velocities and the net effect is a simple phase shift. Due to the very stable nature of solitonic structures non-linear systems exhibiting this kind of solutions may take some time to equilibriate. We will say more about solitons later, for the moment we concentrate on field theories in general. In this chapter we explain and report some basic facts behind equilibration and relaxation properties of physical systems. We start with basic facts of hamiltonian field theories. Past studies focused mainly on analytic initial conditions for which the power spectrum on not analytic at $k = 0$ in momentum space. Starting form such initial conditions it is possible to study time scales involved in the relaxation towars equilibirum. After, we go through the thermodynamic limit Bethe equations, showing how these can be employed to compute thermodynamic quantities in classical theories and discussing problems and implications for the non-equilibirum situation. An important by-product of the TBA formalism, will be the introduction of a fundamental quantity, the *root density*: this function completely determine the system at hand and its thermodynamic behavior at zero temperature. Actually, the root density is the quantity which encodes all the information about the initial state of the system. In Chapter 3 we will link the classical version of the root density to the generating function of all conservation laws in the NLSE. A first thing to say is about nomenclature: for a generic system the approach to equilibrium is well understood and we call this process *thermalization*. For integrable models we rather talk about *relaxation* or *equilibration*.

The peculiar behavior of integrable systems will emerge from the different dynamical behavior imposed by the extensive set of conservation laws present in these theories.

2.1 Relaxation in Integrable Field Theories

Before starting, let's say more about the definitions needed to compute stationary state properties of a classical system. We consider only *local*and *extensive* functionals of the generic fundamental field ϕ (it can be whatever, even a multiplet of fields). What it is usually argued [58] is that during the measurement process of a certain quantity, observables still continue to change so that what we measure is actually an average over time. Further, the time interval we average over is supposed to be very large with respect to that of microscopic variations. If $\mathcal{F}[\phi(x,t)]$ is a generic observable we define,

Definition 2. *(Spatial Average)*

$$\bar{\mathcal{F}}(t) = \lim_{L\to+\infty} \frac{1}{L}\int_0^L \mathrm{d}x \mathcal{F}[\phi(x,t)] \tag{2.1}$$

Definition 3. *(Time Average)*

$$\langle\mathcal{F}\rangle(t) = \frac{1}{t}\int_0^t \mathrm{d}\tau \bar{\mathcal{F}}(\tau) \tag{2.2}$$

Definition 4. *(Classical Dynamical Average)*

$$\langle\mathcal{F}\rangle_{CDA} = \lim_{t\to\infty} \langle\mathcal{F}\rangle(t) \tag{2.3}$$

In Chapter 1 we have introduced probability distributions on field and mentioned that they can be used to predict steady state properties. This is precisely the ergodic hypotesis. For non integrable theories it reads,

$$\langle O\rangle_{CDA} = \langle O\rangle_{\beta} \tag{2.4}$$

while its generalization for integrable ones is,

$$\langle O\rangle_{CDA} = \langle O\rangle_{\underline{\mu}} \tag{2.5}$$

To show that for integrable theories the second statement is indeed correct we can take the easiest integrable theory we know: again the free model. Consider the non relativistic free hamiltonian,

$$H = \int \mathrm{d}x|\partial_x\psi|^2 \tag{2.6}$$

As we know the filling fraction $|A(k)|^2$ is constant in time so that its value is fixed once the initial conditions are given. This means that,

$$\langle|A(k)|^2\rangle_{CDA} = \int \mathrm{d}k|A(k)|^2 \tag{2.7}$$

explicitely depends on the initial conditions. This is in contrast with the assumptions of classical statistical mechanics for which the knowledge of initial conditions is lost and we can safely use a Gibbs ensemble to describe steady state properties. Obviously, the thermal average is,

$$\langle A(k)\bar{A}(q)\rangle_\beta = \frac{\delta(k-q)}{\beta\epsilon(k)} \tag{2.8}$$

which is different from the classical dynamical average. This very simple argument shows the inadequacy of thermal states to describe expectation values of local observables in integrable theories. It is clear that an ensemble able to reproduce the correct behavior of the steady state should have memory of the initial conditions. To check that the GGE average $\langle\circ\rangle_{\underline{\mu}}$ correctly describe the steady state we note that conserved charges were found to be of the form,

$$\mathcal{Q}_n = \int \mathrm{d}k q_n(k)|A(k)|^2 \tag{2.9}$$

with $q_n(k) = k^n$ being the charge eigenvalues. This mean that the GGE is diagonal in momentum space (dropping the superscript GGE) and we describe it by an effective lagrange multiplier[1] $\eta(k)$ [52],

$$\mathcal{Z}(\eta) = \int \mathcal{D}(A,\bar{A})e^{-\int \mathrm{d}k\eta(k)|A(k)|^2} \tag{2.10}$$

This is called *Momentum Space GGE*. We easily find,

$$\langle A(k)\bar{A}(q)\rangle_\eta = \frac{\delta(k-q)}{\eta(k)} \tag{2.11}$$

with the remaining expectations between fields vanishing. Thus, from the GGE we recover the correct expectation value,

$$\langle\psi\bar{\psi}\rangle_{CDA} = \langle\psi\bar{\psi}\rangle_\eta \tag{2.12}$$

Using Wick's therem the computation can be extended to the computation of arbitrary powers of $\psi\bar{\psi}$.

2.2 Classical Quenches

In the Introduction we have already said that the out-of-equilibrium dynamics of many body one dimensional systems has been probed in the contex of the quantum quench protocol [3–5]. The idea of the quench is simple. The dynamics of a quantum system is ruled by a Hamiltonian $H(g)$ which depends on some parameter g. For instance, g may be the interaction strength or the external magnetic field. At $t < 0$, one prepares the system in an eigenstate $|\psi_0\rangle$ of $H(g_0)$, displaying at the same time well defined thermodynamic properties, as extensivity and the cluster decomposition property. At $t = 0$, the quench parameter is suddenly changed to a new value $g \neq g_0$. In this way

[1] Note tha the series is in principle $\sum_{n=0}^{+\infty}\mu_n q_n(k) = \eta(k)$ but the series is divergent.

the time evolution of the state is non trivial,

$$|\psi(t)\rangle = e^{-iHt}\,|\psi_0\rangle \tag{2.13}$$

due to quantum coherence phenomena. It is clear that since the dynamics is unitary the system cannot equilibrate. Neverthless, it is possible to have local thermodynamic equilibrium at level of finite subsystems. The interested reader may find a more comprehensive discussion on requirements and definitions in the references. In the classical case, the analog of the quantum quench may be though as follows. A classical field evolves according to a non linear differential equation derived from the Hamiltonian $H(g)$ with a certain coupling paramenter g: this will be our quench parameter. In this case we say that the evolution is ruled by $H(g)$. Since there is no ground state in a classical field theory, we choose an initial field configuration from a thermal state or a GGE of $H(g_0)$ and follow the time evolution according to $H(g)$. This protocol is the classical analog of the homogeneous quantum quench. Recently, a similar problem has been analyzed in the contex of the Inverse Scattering Method[59], a topic of the next Chapter. In this paper the auhors, introduce a particular mapping, called the *quench map*, used to construct post-qeunch solutions given pre-quench initial conditions. Specifically, they consider solitonic solutions as initial conditions. In the quantum setting, besides pure states, one could consider physical density matrices which are stationary with respect to the pre-quench hamiltonian, such a thermal states. In this case, on top of the quantm uncertainty, expectation values encode classical statistical noise as well. With the classical real, the analogue of density matrices are statistical ensembles. Therefore, we randomly draw initial conditions from a given statistical ensemble and deterministically evole them. As an example of classical quench we can consider quenching of the mass in the free relativistic field theory in a thermal state at inverse temperature β. For $t < 0$ the Hamiltonian reads,

$$H_{t<0} = \frac{1}{2}\int \mathrm{d}x \left\{\pi^2 + (\partial_x\phi)^2 + m^2\phi^2\right\} \tag{2.14}$$

$$\{B(k), \bar{B}(q)\} = i\delta(k-q) \tag{2.15}$$

with all the other commuator vanishing. At $t = 0$ we suddenly quench the mass to a new value and, for $t > 0$, the hamiltonian becomes,

$$H_{t>0} = \frac{1}{2}\int \mathrm{d}x \left\{\pi^2 + (\partial_x\phi)^2 + M^2\phi^2\right\} \tag{2.16}$$

because the reality condition of $\phi(x), \pi(x)$ imposes $\bar{\phi}(k) = \phi(-k), \bar{\pi}(k) = \pi(-k)$. For $t < 0$ the field can be expandend in normal modes,

$$\phi_{t<0}(k,t) = \frac{1}{\sqrt{2\epsilon_0(k)}}\left\{B(k)e^{-i\epsilon_0(k)t} + \bar{B}(k)e^{i\epsilon_0(k)t}\right\} \tag{2.17}$$

$$\pi_{t<0}(k,t) = i\sqrt{\frac{\epsilon_0(k)}{2}}\left\{\bar{B}(k)e^{-i\epsilon_0(k)t} - B(k)e^{i\epsilon_0(k)t}\right\} \tag{2.18}$$

where $\epsilon_0(k) = \sqrt{k^2 + m^2}$ is the pre-quench dispersion relation and of course,

$$\langle B(k)\bar{B}(q)\rangle = \frac{\delta(k-q)}{\beta\epsilon_0(k)} \tag{2.19}$$

Also at $t = 0$,

$$B(k) = \sqrt{\frac{\epsilon_0(k)}{2}}\phi(k) + i\frac{\pi(k)}{\sqrt{2\epsilon_0(k)}} \tag{2.20}$$

$$\bar{B}(k) = \sqrt{\frac{\epsilon_0(k)}{2}}\phi(-k) - i\frac{\pi(-k)}{\sqrt{2\epsilon_0(k)}} \tag{2.21}$$

An analogous expansion holds at $t > 0$,

$$\phi_{t>0}(k,t) = \frac{1}{\sqrt{2\epsilon(k)}}\left\{A(k)e^{-i\epsilon(k)t} + \bar{A}(k)e^{i\epsilon(k)t}\right\} \tag{2.22}$$

$$\pi_{t>0}(k,t) = i\sqrt{\frac{\epsilon(k)}{2}}\left\{\bar{A}(k)e^{-i\epsilon(k)t} - A(k)e^{i\epsilon(k)t}\right\} \tag{2.23}$$

where here $\epsilon(k) = \sqrt{k^2 + M^2}$ is the post-quench dispersion relation. Invoking the continuity of the fields at $t = 0$ we find a relation between mode occupations $A, \bar{A}, B, \bar{B}$. Indeed, inverting the equations above at $t = 0$ gives,

$$B(k) = \sqrt{\frac{\epsilon(k)}{2}}\phi(k) + i\frac{\pi(k)}{\sqrt{2\epsilon(k)}} = f_+(k)A(k) + f_-(k)\bar{A}(k) \tag{2.24}$$

$$\bar{B}(-k) = \sqrt{\frac{\epsilon(k)}{2}}\phi(k) - i\frac{\pi(k)}{\sqrt{2\epsilon(k)}} = f_-(k)A(k) + f_+(k)\bar{A}(k) \tag{2.25}$$

where we have used the reality condition of $\phi(x), \pi(x)$ to get $\bar{\phi}(k) = \phi(-k), \bar{\pi}(k) = \pi(-k)$ and defined the even and real functions,

$$f_\pm(k) = \frac{1}{2}\left\{\sqrt{\frac{\epsilon(k)}{\epsilon_0(k)}} \pm \sqrt{\frac{\epsilon_0(k)}{\epsilon(k)}}\right\} \tag{2.26}$$

In matrix notation,

$$\begin{pmatrix} B(k) \\ \bar{B}(-k) \end{pmatrix} = \begin{pmatrix} f_+(k) & f_-(k) \\ f_-(k) & f_+(k) \end{pmatrix}\begin{pmatrix} A(k) \\ \bar{A}(k) \end{pmatrix} \tag{2.27}$$

We obtain $A(k), \bar{A}(k)$ in term of old ones,

$$A(k) = f_+(k)B(k) + f_-(k)\bar{B}(-k) \tag{2.28}$$

$$\bar{A}(k) = f_+(k)\bar{B}(k) + f_-(k)B(-k) \tag{2.29}$$

Now we compute correlators,

$$\langle A(k)\bar{A}(q)\rangle = (f_+(k)f_+(q) + f_-(k)f_-(q))\frac{\delta(k-q)}{\beta\epsilon_0(k)} \tag{2.30}$$

$$\langle A(k)\bar{A}(q)\rangle = \frac{1}{2}(f_+(k)f_-(q) + f_-(k)f_+(q))\frac{\delta(k-q)}{\beta\epsilon_0(k)} \tag{2.31}$$

Here we used known Poisson brackets between Bs, the knowledge of the initial state $\langle B(k)\bar{B}(q)\rangle = \frac{\delta(k-q}{\beta\epsilon_0(k)}$ and $\delta(2x) = \frac{1}{2}\delta(x)$. The two point function is given by,

$$\langle\phi(x,t)\phi(0,t)\rangle = \frac{1}{2\pi}\int \frac{\mathrm{d}k\mathrm{d}q}{\sqrt{2\epsilon(k)}\sqrt{2\epsilon(q)}} \tag{2.32}$$

$$\Bigg\{\langle A(k)\bar{A}(q)\rangle e^{ikx - i\epsilon(k)t + i\epsilon(q)t} \tag{2.33}$$

$$+ \langle A(q)\bar{A}(k)\rangle e^{-ikx + i\epsilon(k)t - i\epsilon(q)t} \tag{2.34}$$

$$+ \langle A(q)A(k)\rangle e^{ikx - i\epsilon(k)t - i\epsilon(q)t} \tag{2.35}$$

$$+ \langle \bar{A}(q)\bar{A}(k)\rangle e^{-ikx + i\epsilon(k)t + i\epsilon(q)t}\Bigg\} \tag{2.36}$$

Putting everything togheter,

$$\langle\phi(x,t)\phi(0,t)\rangle = \frac{1}{2\pi\beta}\int \frac{\mathrm{d}k}{\epsilon_0(k)\epsilon(k)} e^{ikx}\Bigg\{((f_+(k))^2 + (f_-(k))^2) \tag{2.37}$$

$$+ (f_+(k)f_-(k))\cos(2\epsilon(k)t)\Bigg\} \tag{2.38}$$

In the limit $t \to +\infty$ the oscillatory factor vanishes and we get,

$$G(x) = \lim_{t\to+\infty}\langle\phi(x,t)\phi(0,t)\rangle = \frac{1}{2\pi\beta}\int \frac{\mathrm{d}k}{\epsilon_0(k)\epsilon(k)} e^{ikx}((f_+(k))^2 + (f_-(k))^2) \tag{2.39}$$

$$= \frac{1}{4\pi\beta}\int \mathrm{d}k e^{ikx}\frac{2k^2 + M^2 + m^2}{(k^2+m^2)(k^2+M^2)} \tag{2.40}$$

$$= \frac{1}{4\beta\sqrt{2}}\left[\frac{1}{M}e^{-M|x|} + \frac{1}{m}e^{-m|x|}\right] \tag{2.41}$$

This computation shows that the large time behavior of the two point function retains memory of its initial conditions preventing it to be obtained by a Gibbs distribution. Assuming the final state to be thermal,

$$\langle A(k)\bar{A}(k)\rangle = \frac{\delta(k-q)}{\beta\epsilon(k)} \tag{2.42}$$

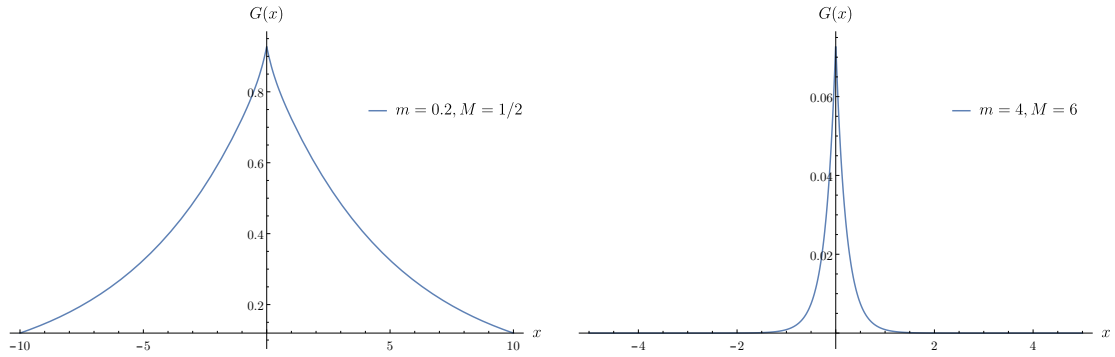

FIGURE 2.1: One point function (2.41) for different quenches at $\beta = 1$.

the correlator is,

$$\lim_{t\to+\infty}\langle\phi(x,t)\phi(0,t)\rangle = \frac{1}{\pi\beta}\int\frac{\mathrm{d}k}{(\epsilon(k))^2}e^{ikx} = \frac{e^{-M|x|}}{2\beta M\sqrt{2}} \tag{2.43}$$

which of course is different from the previous one. The expression is consistent with (2.41) since if $m = M$ there is no quench, the system remains in the initial, thermal, state and the correlator must be given by a Gibbs average. This is basically what happens in continuous free field theories. Despite this example reveals many interesting aspects of the classical quench we want to study a fully interacting model.

We do this quenching the Non Linear Schrodinger system initializing it in different initial states. Recall the NLSE Hamiltonian,

$$H_{NLS}(g) = \int_0^L \mathrm{d}x\left\{|\partial_x\psi(x)|^2 + g|\psi|^4\right\} \tag{2.44}$$

Specifically, in one case pre-quench initial conditions are drawn from thermal states of the Hamiltonian,

$$H_{free} = \int_0^L \mathrm{d}x\left\{|\partial_x\psi(x)|^2 - \mu|\psi(x)|^2\right\} \tag{2.45}$$

where μ is the chemical potential. Note, that since we have a bosonic theory $\mu < 0$. This initial state allows to fix the initial density,

$$n = \frac{\langle N\rangle}{L} = \frac{1}{L}\int_0^L dx\langle\bar{\psi}(x)\psi(x)\rangle = \frac{1}{L}\int_0^L dx\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\frac{1}{\beta\epsilon(k)} = \frac{1}{\beta}\sqrt{\frac{-1}{2\mu}} \tag{2.46}$$

where the dispersion relation is given in (1.127). The higer the density the stronger is the interaction since there are more particles in the system. After, we evolve according to the interacting Hamiltonian $H_{NLS}(g)$ with non zero interaction. In the second case,

we consider an initial GGE for which,

$$\langle \bar{\psi}(k)\psi(q)\rangle = \delta(k-q)e^{-\beta\epsilon(k)} \tag{2.47}$$

where again $\epsilon(k)$ is given in (1.127). Here the density is,

$$n = \frac{e^{\beta\mu}}{2\sqrt{\pi\beta}} \tag{2.48}$$

It should be stressed that β has not the meaning of temperature here. We use this GGE mainly to exponentially suppress UV modes present in the theory. This last fact will be best explained later on. After an initial condition has been drawn and again we evolve according to the interacting hamiltonian. The problem is to compute steady-state properties of the statistical system, that is the $t \to +\infty$ limit of expectation values of observables, analogously to what we did analytically in the free theory. Of course, more general initial states are possible.

2.3 Thermalization Time Scales

The understanding of the time scales involved in the relaxation process is an important task. This can be accomplished when the field is initialized to particular initial conditions. It is imporant to stress that the choice of initial conditions for which the following analysis can be done is different from the previously introduced quenching protocol: here we take an analytical initial field while before we took ensembles of initial conditions. Notice that, in the case of thermal ensembles $|A(k)|^2 \sim k^{-2}$ so that these fields are not analytical. In what follows every field will be periodic in the spatial varible with period L and the time will be taken in $[0, +\infty]$. The former condition ensures that energy is extensive when eventually we take $L \to +\infty$. Consider a $1+1$ relativistic local Langrangian containing a single bosoninc excitation with bare mass m and bare coupling g,

$$\mathcal{L}_R = \frac{1}{2}(\partial_{x^0}\phi)^2 - \frac{1}{2}(\partial_{x^1}\phi)^2 - m^2 V_R(g\phi) \tag{2.49}$$

and Hamiltonin,

$$\mathcal{H}_R = \frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_{x^1}\phi)^2 + m^2 V_R(g\phi) \tag{2.50}$$

keeping for the moment the potential as general as possible. At the classical level it is possible to rescale coordinates and fields, in order to get dimensionless Langrangian, Hamiltonian and equations of motion. Rescaling,

$$\begin{aligned} t &= mx^0 \\ x &= mx^1 \\ \varphi(x,t) &= g\phi(x^0,x^1) \\ \pi(x,t) &= \phi_t(x,t) \end{aligned}$$

we find,

$$\mathcal{L}_R = \frac{m}{g^2}\hat{\mathcal{L}}_R \tag{2.51}$$

$$\mathcal{H}_R = \frac{m}{g^2}\hat{\mathcal{H}}_R \tag{2.52}$$

with,

$$\hat{\mathcal{L}}_R = \frac{1}{2}(\partial_x\varphi)^2 - \frac{1}{2}(\partial_x\varphi)^2 - V_R(\varphi) \tag{2.53}$$

$$\hat{\mathcal{H}}_R = \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x\varphi)^2 + V_R(\varphi) \tag{2.54}$$

where the subscritp stands for "relativistic". In this way equations of motion are,

$$\partial_t^2\varphi - \partial_x^2\varphi + \frac{\mathrm{d}V_R(\varphi)}{\mathrm{d}\varphi} = 0 \tag{2.55}$$

In particular,

$$V_R(\phi) = (1 - \cosh(\varphi)) \tag{2.56}$$

gives the ShG Langrangian while,

$$V_R(\phi) = \varphi^4 \tag{2.57}$$

the Ginzburg-Landau model. If we consider a general non-relativistic field theory we will have,

$$\mathcal{L}_{NR} = \frac{i}{2}\left(\bar{\psi}\partial_{x^0}\psi - \psi\partial_{x^0}\bar{\psi}\right) - \frac{1}{2m}|\partial_{x^1}\psi|^2 - V_{NR}(\bar{\psi},\psi) \tag{2.58}$$

and

$$\mathcal{H}_{NR} = \frac{1}{2m}|\partial_{x^1}\psi|^2 + V_{NR}(\bar{\psi},\psi) \tag{2.59}$$

Field equations follow from a variation with respect to ψ and $\bar{\psi}$ taken as independent variables,

$$i\partial_t\psi = -\partial_x^2\psi + \frac{\delta V(\psi,\bar{\psi})}{\delta\bar{\psi}} \tag{2.60}$$

We note that the fundamental field of a non-relativistic theory must necessarily be complex. Indeed, a complex field allows to construct $U(1)$ invariant theory, a necessary requirement in the non-relativistic world, since particle production is forbidden and number of particles must be conserved. Thus,

$$N = \int_0^L \mathrm{d}x\bar{\psi}\psi \tag{2.61}$$

si always conserved. Since the theory is local and translation invariant, the total momentum of the field is conserved too,

$$P = -\int_0^L \mathrm{d}x \pi \partial_x \psi = -i \int_0^L \bar{\psi} \partial_x \psi \tag{2.62}$$

We will take as a basis the NLSE potentail, namely,

$$V_{NR}(\bar{\psi}, \psi) = g|\psi|^4 \tag{2.63}$$

since it seems that despite the variety of integrable relativistic models exhibiting exciting and rather different behaviors, theories attainable by these after suitable non-relativistic limit and requiring integrability seem to be confined to the above quartic potential [60]. If we rescale,

$$\begin{aligned} \psi &= \alpha \Psi \\ x^1 &= \delta x \\ x^0 &= \beta t \end{aligned}$$

we find that, choosing units such that the field is adimensional, that is $\alpha = 1$, it follows that $\beta = g^{-1}$, $\delta = (2mg)^{-1/2}$ and,

$$\mathcal{L}_{NR} = \left(\frac{g}{2m}\right)^{1/2} \hat{\mathcal{L}}_{NR} \tag{2.64}$$

$$\mathcal{H}_{NR} = \left(\frac{g}{2m}\right)^{1/2} \hat{\mathcal{H}}_{NR} \tag{2.65}$$

with a dimensionless Hamiltonian and Langrangian,

$$\hat{\mathcal{L}}_{NR} = \frac{i}{2}\left(\bar{\Psi}\partial_t \Psi - \Psi \partial_t \bar{\Psi}\right) - |\partial_x \Psi|^2 - |\Psi|^4 \tag{2.66}$$

$$\hat{\mathcal{H}}_{NR} = |\partial_x \Psi|^2 + |\Psi|^4 \tag{2.67}$$

The dimensionless equations of motion are,

$$i\partial_t \Psi = -\partial_x^2 \Psi + 2|\Psi|^2 \Psi \tag{2.68}$$

This means that equations of motion do not care about the actual value of the parameters, as it is always possible to reasbsorb them. We will see in a moment that this is not true when we do statistics on fields, *i.e.* we introduce a probability measure on the space of fields. Since our interest is in the understanding of emergent properties of the steady state, we are automatically asked to explore the $t = \infty$ limit of the solutions to equations of motion. For a smooth initial conditions of initial value problems considered in this thesis it is known that a unique solution exists (citation). If the system is supposed to follow an ergodic dynamics, as times goes, an equipartition between modes should appear and the potential energy should entirely be transferred the kinetic part. Thus, for very large k we would have,

$$|\tilde{\phi}(k,t)|^2 = const.k^{-2} \tag{2.69}$$

because only the laplacian term would be important. This behavior of the Fourier components of the field corresponds to a non-analytic behavior of $\phi(x,t)$. From this, one can argue that complete *thermalization* can occur only at infinite time and the field has to develop singlularities in the complex x-plane such that as $t \to \infty$ they coalesce the real axis. Indeed, thanks to periodicity of the field we can write,

$$\tilde{\varphi}(k,t) = \int_{-L/2}^{L/2} \mathrm{d}x e^{-ikx} \varphi(x,t) \tag{2.70}$$

As $L \to +\infty$,writing $z = x + iy$, we can apply residue theorem,

$$\tilde{\varphi}(k,t) = 2\pi i \sum_j R_j e^{-ikx_j - |ky_j|} \tag{2.71}$$

This mean that the power spectrum, for large k, behaves as,

$$W(k,t) = |\tilde{\varphi}(k,t)|^2 \sim e^{-2k|y_s(t)|} \tag{2.72}$$

where $y_S(t)$ is the imaginary part of the pole nearest to the real line (other terms go to zero faster). This argument has been applied in [61, 62] and also numerically verified. They estimated $y_S(t)$ considering the complexification of the evolution equation of the φ^4 model and viewing the solution as a function of the initial condition $\varphi = \varphi(\varphi_0, t)$. Since our study focus on the NLSE, we try the same ideas on it, making comparisons with know predictions in other models. To our knowledge this is the first application to a non-relativistic field theory. Moreover, it should be remarked that an application of the method to integrable theories is not expected to give reliable results, since strict thermalization does not occurr. The natural question is about the differences, if there are any, between relativistic and non-relativistic models, integrable and not. The study of the appearence of singularities in the field gives informations about time scale involved in the relaxation process towards equilibrium. We first assume that the laplacian term is negligible. A situation where this is true is that considered by [57, 61, 62] for the φ^4-theory and partly by [9], for the ShG model. In these works the initial configuration is taken to be,

$$\Psi(x, t=0) = A\cos(k_0 x) \tag{2.73}$$

thus,

$$\max|\partial_x^2 A\cos(k_0 x)| / \max|V(A\cos(k_0 x))| = \frac{Ak_0^2}{V(A)} \ll 1 \tag{2.74}$$

for sufficiently large amplitudes or small wave vectors and polynomial potentials of degree greater then 1 so that for short times we can neglect the laplacian term. Separating real and imaginary part of (2.68) the system of equations we get is,

$$\begin{cases} \dot{\Psi}_r &= 2|\Psi|^2\Psi_i \\ \dot{\Psi}_i &= -2|\Psi|^2\Psi_r \end{cases} \tag{2.75}$$

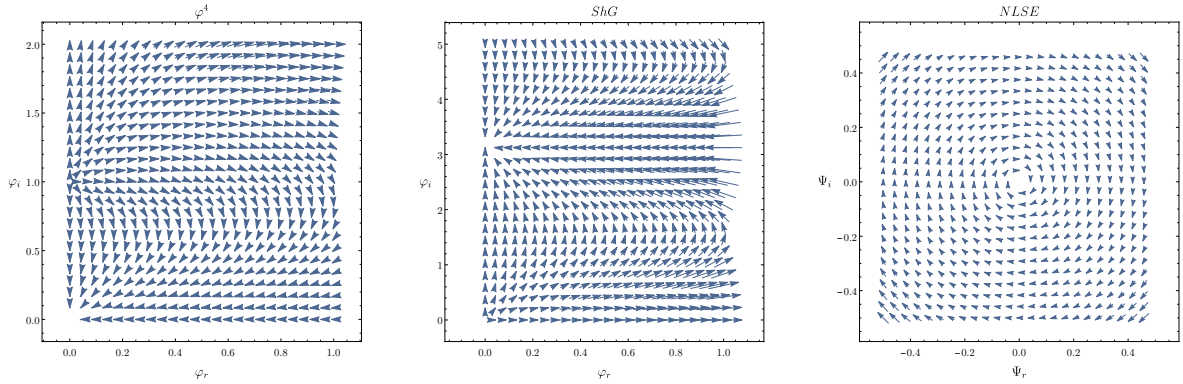

FIGURE 2.2: Trajectories of the system for the real and imaginary parts of the field in the short time regime. It is clear that for the φ^4-theory and ShG model the ifinity is a saddle point that can be reached in a finite amount of time. Interestingly NSLE does not show the same behavior and the system oscillates indefinitely.

where Ψ_r and Ψ_i denote real and imaginar part respectively and the dot the time derivative. The system immediately integrates to,

$$\Psi_r(\Psi_i) = \pm\sqrt{C - \Psi_i^2} \tag{2.76}$$

Thus, trajectories are stable circles around the origin as shown in Fig. 2.2. and the system (2.75) becomes,

$$\begin{cases} \dot{\Psi}_r &= 2\Psi_i \\ \dot{\Psi}_i &= -2\Psi_r \end{cases} \tag{2.77}$$

which has solutions,

$$\Psi_r(t) = A_r \cos(2t + B_r) \tag{2.78}$$
$$\Psi_i(t) = A_i \cos(2t + B_i) \tag{2.79}$$

The field oscillates indefinitely and does not diverge at infinity, thus no singularities can be produced. Here, the argument seems to give an even stronger reason for thermalization not to occur. This behavior is independent on the coupling g and the mass m, mirroring the exact integrability of the model for any value of these parameters. Comparison with the other integrable field theory, the ShG, deserves some comment. While the φ^4-theory is expected to thermalize at a certain point, the ShG is integrable. In these cases the argument applies and from Fig. 2.2 it can be seen the the field can go at infinite in a finite amount of time. The estimate for the location of the pole nearest to the real axis in the short time scale presented in [9, 62] is,

$$y_s^{ShG}(t) = \frac{1}{k_0} \log\log \frac{\sqrt{2}}{t} \tag{2.80}$$
$$y_s^{\varphi^4}(t) = -\frac{1}{k_0} \log \frac{At}{\sqrt{12}} \tag{2.81}$$

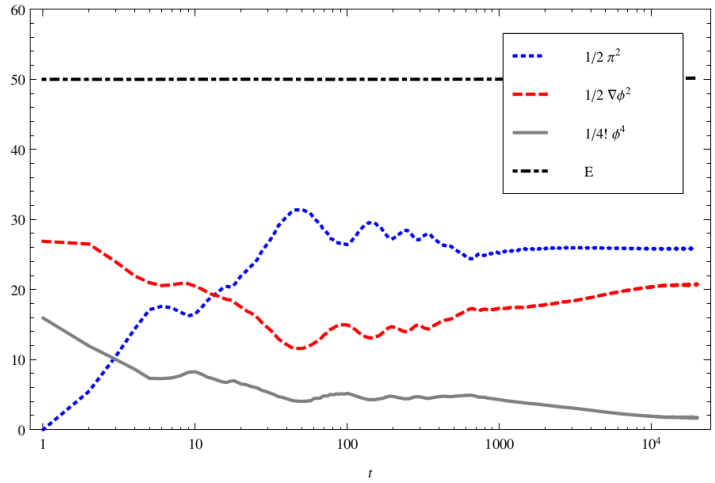

FIGURE 2.3: Time evolution of observables in the φ^4-theory. Time axis in log scale. Figure taken from Ref. [9].

anyhow confirming the extremely slow dynamics of the ShG toward relaxation with respect to the φ^4-theory. All the past studies related to the large time behavior of systems descibed by field theories have been an imporant thing in common: they all considered a given initial condition whose energy spectrum was strongly peaked in the IR. Remember that we have said the solitonic excitations may be present and so relaxation toward mode mixing is expected to be slow. The situation is even worst if the theory is integrable as the above estimates confirm. In these cases, given a an initial field configuration it is possible to distinguish between three different time scales during the relaxation towards equilibrium, as depicted in Fig. 2.3. For it is has been said about the NLSE model, times needed to observe a sort of equilibration are extremely long.

- A *short time scale* t_S during which the laplacian term is less important (see (2.74)) and the already occupied modes start to mix producing a quasi-thermal state. This is the time at which laplacian and potential terms become of the same order of magnitude.

- An *intermediate time scale* t_I at which the laplacian is much larger the the interaction terms. This time scale sets the the brunch point where the dynamics originially dominated by interactions becomes dominated by kinetic energy terms.

- A *large time scale* T_L where the equilibration takes place. The large time mechanims responsible of mode mixing has been called *drop-by drop phenomenon*, mirroring the slow dynamics. Higher and higer modes get slowly activated, due to the fact that interctions are not able to efficiently transfer energy among modes anymore. In this regime the theory is almost free.

With reference to Fig. 2.3 the time scale t_S can be identified with the point at which the gradient term and the interaction cross; the intermediate time scale t_I as the point where the curves change their concavity; the time t_L is instead when curves reach their asymptotic values. Note also that while for non-integrable field theories one could be interested in the observation of the energy cascade towards the UV, thus initializing the field in IR make sense, for integrable theories the dynamics of the modes occupation

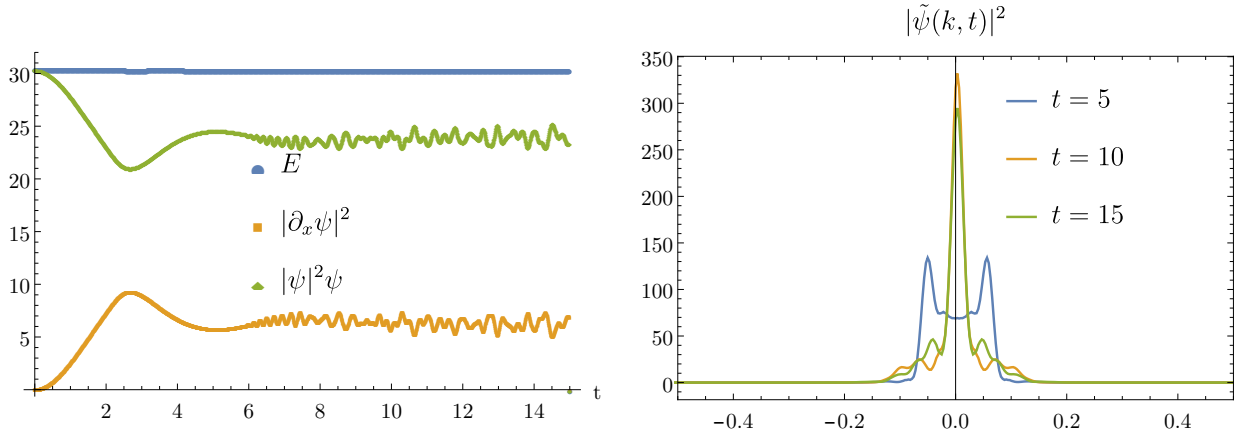

FIGURE 2.4: Time evolution of observables in the NLSE model (left) and power spectrum (right). The initial configuration is $\psi(x,0) = 3\sin(\frac{2\pi}{L}x)$.

is expected not exhibit particularly interesting behavior. Indeed, see Fig. 2.4,integrable theories like the ShG model and the NLSE are diagoanlizable by means of action-angle variables. Mode occupations are associated to action variables, whose dynamics is essentialy that of a free system.

2.4 On the Thermodynamic Limit of the Lieb-Liniger Model

2.4.1 Thermodynamic Bethe Ansatz at Zero Temperature

In the past section we have analyzed field theories in general. We now go back to the Lieb-Liniger model and study its thermodynamics. This will serve as a basis to do the same for classical theories as will be seen in Chapter 3. The physical picture for a systems with a large number of particles is that in the thermodinamic limit the roots of the Bethe equations densely fill some interval of the real line.Indeed, while the I_j's are equally spaced on a uniform lattice in the ground state, the quasi-momenta in general are not. Thus, we expect there is a distribution $\rho(k)$ such that $L\rho(k)\mathrm{d}k$ is the number of k's in $[k, k+\mathrm{d}k]$. As a notation, we write $\lim_{TH}$ meaning the thermodynamic limit, that is $N, L \to +\infty$ with N/L kept fixed. Define the density of quasi-momenta,

$$\rho(k_j) = \lim_{TH} \frac{1}{L(k_j - k_{j-1})} > 0 \tag{2.82}$$

We define also,

$$y(k) = Lk - \sum_{l=1}^{N} \theta(k - k_l) \tag{2.83}$$

This function is monotonically increasing and is such that if k_j is a root,

$$y(k_j) = \frac{2\pi I_j}{L} \tag{2.84}$$

It follows that,

$$\frac{d}{dk}y(k) = \lim_{TH} \frac{y(k_{j+1}) - y(k_j)}{k_{j+1} - k_j} = 2\pi\rho(k) \tag{2.85}$$

because in the ground state $I_{j+1} - I_j = 1$. Thus, in the thermodynamic limit, we can substitute $\frac{1}{L}\sum_{i=1}^{N} \mapsto \int \rho(k)\mathrm{d}k$ and the Bethe equations (1.160) get transformed into a single linear integral equation,

$$\rho(k) = \frac{1}{2\pi} + \frac{1}{2\pi}\int_{k_{min}}^{k_{max}} \varphi(k-q)\rho(q)\mathrm{d}q \tag{2.86}$$

with the Lieb-Liniger kernel defined by,

$$\varphi(k) = -\frac{d}{dk}\theta(k) = \frac{2g}{g^2 + k^2} \tag{2.87}$$

Within this formalism, the ground state comes from a symmetric support $k_{min} = -k_{max} \equiv q$. This integration limit can be fixed in terms of the density by,

$$n = \lim_{TH} N/L = \int_{-q}^{q} \rho(k)\mathrm{d}k \tag{2.88}$$

The ground state energy and momentum are thus,

$$e = \lim_{TH} E/L = \int_{-q}^{q} k^2\rho(k)\mathrm{d}k \tag{2.89}$$

$$p = \lim_{TH} P/L = \int_{-q}^{q} k\rho(k)\mathrm{d}k \tag{2.90}$$

Since the ground state is expected to have zero momentum we can conclude that the root density ρ must be an even function of its argument.

2.4.2 Elementary Excitations at $T = 0$

In one dimensional systems like the LL model there are two kinds of elementary excitations, called type I excitations and type II excitations. Actually there can be more complicated kinds of excitations but these two are the building blocks: any other type can be generated by a finite combination of these. We have seen that a state of the system at finite N and L is specified by a set of integers or semi-integers I_j which in turn determine a set of roots of the Bethe equations (1.160). Also, the ground state is characterized by the following set of integers,

$$I_j = \frac{-N+1}{2} + j \quad j \in \{1, \ldots, N\}$$

In the thermodynamic limit we have a distribution of roots ρ supported in $[-q, q]$. Excitations are,

1. **Type I excitations** : we add a particle ($\Delta N = 1$) in a state with $|k_p| > q$.
2. **Type II excitations** : we remove a particle ($\equiv$add a hole, ($\Delta N = -1$)) in a state with $|k_h| < q$.

As an example of non-elementary excitation we can form by these two, one can cosider adding a particle and adding a hole: clearly, $\Delta N = 0$ and the net effect is the same as if a particle with $|k| < q$ has been moved ($\equiv$excited) to a state with $|k| > q$. In Fig. 2.5 it possible to visualize these excitations.

Type I Excitations

Let us see focus for sake of clarity on particle excitations. Adding a particle in a state $|k_p| > q$ means shifting the quantum number at the edge of the Brilluoin zone by an integer m. The new set of integers will be,

$$\{I'\} = \{-\frac{N}{2} - \frac{N}{2} + 1, \ldots, \frac{N}{2} + m\} \tag{2.91}$$

The total momentum will be,

$$P = \frac{2\pi m}{L} \tag{2.92}$$

This is called *dressed* momentum in contrast to the *bare* momentum k_p of the added particle. The system rearranges itself as a whole: the original set of roots $\{k_1, \ldots, k_N\}$ is shifted to a new set $\{k'_1, \ldots, k'_N, k_p\}$, also solution of (1.160) but with the different set of integers (2.91). The total momentum on the ground state was found to be 0. Here is like we add a particle and the center of mass has total momentum given by (2.92). From these considerations we can compute the change in the energy and momentum of each particle due to the introduction of the new particle. This lead to a new set of integral equation characterizing the excitation. If k'_j is the new quasi-momenta after the introduction of the new particle and k_j is the old one, defining $\Delta k_j = k'_j - k_j$, we can

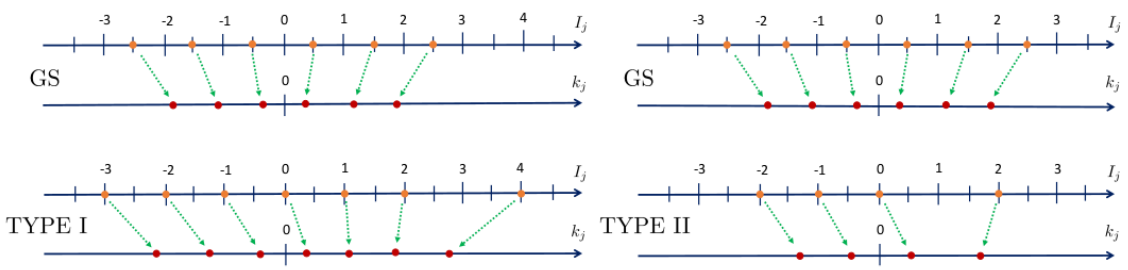

FIGURE 2.5: Excitation spectrum. On the right Type I (particle) excitations. On the left type II (holes) excitations. Cartoon taken from [63].

subtract (1.160) for k'_j and k_j,

$$L\Delta k_j = \pi + \sum_{l=1}^{N} \Big[\theta(k'_j - k'_l) - \theta(k_j - k_l)\Big] + \theta(k'_j - k_p) \tag{2.93}$$

The π appears because $\Delta N = 1$ implies $\Delta I_j = 1/2$. From this relation we see that Δk_j is of order $O(L^{-1})$, thus we can expand the *r.h.s.* to the same order using,

$$\begin{aligned}\theta(k'_j - k'_l) - \theta(k_j - k_l) &= \theta(k_j + \Delta k_j - k_l - \Delta k_l) - \theta(k_j - k_l) \\ &= -\varphi(k_j - k_l)(\Delta k_j - \Delta k_l) + O(L^{-1})\end{aligned}$$

where φ is given in (2.87). After reordering we can write,

$$\Delta k_j \left[1 + \sum_{l=1}^{N} \varphi(k_j - k_l)\right] = \frac{1}{L}\left[\pi + \theta(k_j - k_p)\right] + \frac{1}{L}\sum_{l=1}^{N} \varphi(k_j - k_l)\Delta k_l \tag{2.94}$$

In the thermodynamic limit, afeter using (2.86) in the **l.h.s.** we find an integral equation for the so-called *back-flow* (or *shift*) function,

$$J(k|k_p) - \int_{-q}^{q} \frac{\mathrm{d}\lambda}{2\pi}\varphi(k - \lambda)J(\lambda|k_p) = \frac{1}{2\pi}\tilde{\theta}(k - k_p) \tag{2.95}$$

where $\tilde{\theta}$ is, again, in (1.147). The function $J(k|k_p)$ is defined as,

$$J(k|k_p) = L\Delta k \rho(k) \tag{2.96}$$

and characterizes the excitation. Knowing the back-flow, we can compute the change in the total momentum as,

$$\Delta P(k_p) = k_p + \sum_{i=1}^{N} \Delta k_i = k_p + \int_{-q}^{q} J(k|k_p)\mathrm{d}k \tag{2.97}$$

and the change in the total energy as,

$$\begin{aligned}\Delta e(k_p) &= k_p^2 + \sum_{i=1}^{N}(k_j'^2 - k_j^2) = k_p^2 + \sum_{i=1}^{N}((k_j + \Delta k_j)^2 - k_j^2) \\ &= k_p^2 + \sum_{i=1}^{N}(2k_j\Delta k_j + O(\Delta k_j^2)) = k_p^2 + \int_{-q}^{q} kJ(k|k_p)\mathrm{d}k\end{aligned} \tag{2.98}$$

Again, thinking the system as made by particles moving on a circle and scattering elastically, the back-flow represents the strong correlation between them. Adding a single particle, the whole system gets rearranged. This is what concerns particle excitations.

Type II Excitations

If we remove a particle from a state $|k_h| < q$ by exactly the same reasoning we find an analogous back-flow function which captures the collective rearrangement of the

system. The only difference is that $\Delta N = -1$. The integral equation satisfied by the back-flow is the same as (2.95) but with a minus sign in front of the source term,

$$J(k|k_h) - \int_{-q}^{q} \frac{\mathrm{d}\lambda}{2\pi} \varphi(k-\lambda) J(\lambda|k_h) = -\frac{1}{2\pi}\tilde{\theta}(k-k_h) \tag{2.99}$$

Accordingly, energy and momentum change as,

$$\Delta P(k_h) = -k_h - \int_{-q}^{q} J(k|k_h)\mathrm{d}k \tag{2.100}$$

$$\Delta e(k_h) = -k_h^2 + \int_{-q}^{q} kJ(k|k_h)\mathrm{d}k \tag{2.101}$$

Later on, we will discuss the Thermodynamic Bethe Ansatz at finite temperature, which is nothing more then the generalization of the coordinate Bethe Ansatz we have discussed until now to the finite temperature. It will appear that the expressions for the free energy and other thermodynamic quantities will resemble that of free theories but with an effective dispersion relation, which is obtained from the free one through a proper dressing operation. To each kind of excitation we can associate its particular dispersion relation, called also *dressed* energy, function of the quasi-momenta, and study their properties like scattering or velocity of propagation. For type I excitations the dressed energy can be shown to be quadratic and so are identified with *phonons*. Type II excitations are not simple sound waves and in the small coupling limit can be put in correspondence with dark solitons of the Gross-Pitaevksii equation [63–65].

2.4.3 Thermodynamic Bethe Ansatz at Finite Temperature

The Thermodynamic Bethe Ansatz consists simply in taking the thermodynamic limit of the Bethe equations taking into account particle excitations casued by the non zero temperature. Ref. [30] remains the standard one for the application to the LL model and from a pedagogical point of view the reader is referred to [63]. Also, in the first chapter, we have seen that today it has become known that integrable systems relax towards GGEs which are, of course, rather different from thermal states. This motivates the generalization of the TBA. We will see how the appraoch of [30] is generalized, leading to the so-called *Generalized Bethe Ansatz*: this is nothing more than the Yang-Yang formalism applied to GGEs [66]. The strategy is to minimize the entropy functional under natural constraints which will now explain. We have seen that Bethe equations (2.86), in the thermodynamic limit, are solved by the root density. If the temperature is not zero, in the same way we studied elementary excitations, we have a density of holes ρ_h and a density of roots ρ. However, in this situation, we expect that different choices of these distributions give rise to the same state. To start we consider again the function,

$$y(k) = k - \frac{1}{L}\sum_{i=1}^{N} \theta(k-k_i) \tag{2.102}$$

The states of the system are again ruled by the Bethe equations. By definition, the values k_j such that $y(k_j) = \frac{2\pi I_j}{L}$ are called particles. We generalize this definition by calling possible *states* (or *vacancies*) the valued k_n^s for which y is quantized, $y(k_n^s) = \frac{2\pi n}{L}$. The remaining possibilities are the holes, $\{k_j^h\} = \{k_j^v\} - \{k_j\}$ (as a set identity). We associate to each k corresponding to quantized values a distribution in the thermodynamic limit,

$$\rho(k_j) = \lim_{TH} \frac{1}{L(k_{k+1} - k_j)} \tag{2.103}$$

$$\rho_s(k_j^s) = \lim_{TH} \frac{1}{L(k_{k+1}^s - k_j^v)} \tag{2.104}$$

$$\rho_h(k_j^h) = \lim_{TH} \frac{1}{L(k_{k+1}^h - k_j^h)} \tag{2.105}$$

In this case we find,

$$\frac{\mathrm{d}}{\mathrm{d}k} y(k) = \lim_{TH} \frac{y(k_{j+1}) - y(k_j)}{k_{j+1} - k_j} = \frac{2\pi}{L(k_{j+1} - k_j)} = 2\pi\rho_s(k_j) \tag{2.106}$$

and so,

$$2\pi\rho_s(k) = 1 + \int_{\mathbb{R}} \mathrm{d}\lambda \varphi(k - \lambda)\rho(\lambda) \tag{2.107}$$

Importantly, due to the possibility of arbitrary excitations, the integration here is extended over arbitrary values of k. Note that by definition of vacancies,

$$\rho_s = \rho + \rho_h \tag{2.108}$$

Vacances are nothing more then possible values of quasi-momenta corresponding to quantized values of y, see again Fig. 2.5. Equation (2.107) is not closed since we don't know the hole density so that we cannot solve for the root one. We should use the available information about the statel. Let us consider a thermal state of inverse temperature β,

$$\mathcal{Z} = \sum_{I_1 < I_2 < \cdots < I_N} \exp(-\beta E_N) = \sum_{n_1, \ldots, n_N = 1}^{+\infty} \exp(-\beta E_N) \tag{2.109}$$

where in the second line we have changed variables as $n_j = I_{j+1} - I_j$. In general it is very difficult to compute the energy in terms of the quantum numbers I_j but, once the thermodynamic limit is concerned, its expression is simple in terms of the root density ρ in (2.89). To transform the sum over n_j's into a functional integral over densities (which specify the state in the thermodynamic limit) we first estimate the measure and then the

number of states consistent with given ρ and ρ_h. We have [63],

$$\begin{aligned} n_j &= I_{j+1} - I_j \\ &= \frac{L}{2\pi}\left[y(k_{j+1}) - y(k_j)\right] \\ &= \frac{L}{2\pi}\int_{k_j}^{k_{j+1}} \rho_s(k')\mathrm{d}k' \\ &= \frac{L}{2\pi}\int_{k_j}^{k_j+\frac{1}{L\rho(k_j)}} \rho_s(k')\mathrm{d}k' = \frac{\rho_s(k_j)}{2\pi\rho(k_j)} + O(L^{-1}) \end{aligned}$$

where in the last line we used $k_{j+1} = k_{j+1} - k_j + k_j = k_j + \rho(k_j)/L$. To estimate the number of states consistent with given ρ and ρ_h we calculate the number of ways to distribute $L\rho(k)\mathrm{d}k$ particles and $L\rho_h(k)\mathrm{d}k$ holes in an interbal $\mathrm{d}k$. The entropy of the configuration is thus[2],

$$\mathrm{d}\mathcal{S} = \log\left(\frac{(L\rho\mathrm{d}k + L\rho_h\mathrm{d}k)!}{(L\rho\mathrm{d}k)!(L\rho_h\mathrm{d}k)!}\right) \approx L\left[(\rho+\rho_h)\log(\rho+\rho_h)) - \rho\log\rho - \rho_h\log\rho_h\right]\mathrm{d}k \tag{2.110}$$

This formula comes from the fact that if $n_k = L\rho(k)\mathrm{d}k$ is the number of particle states and $N_k = L\rho_v(k)\mathrm{d}k$ is the number of possible states, the number of ways to distribute n_k objects in N_k possibles boxes is

$$\binom{N_K}{n_K} = \frac{N_K!}{n_k!(N_k-n_k)!}$$

Thus,

$$\begin{aligned} \mathcal{Z} &= C\int \mathcal{D}\left(\frac{\rho_v(k)}{\rho(k)}\right)\delta\left(\int \mathrm{d}k\rho(k) - n\right)\exp\left(\mathcal{S} - \beta L e\right) \\ &= C\int \mathcal{D}\left(\frac{\rho(k)+\rho_h(k)}{\rho(k)}\right)\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}\mu}{2\pi}e^{\mu(\int\rho\mathrm{d}k - n)}\exp\left(\mathcal{S} - \beta L e\right) \\ &= C\int \mathrm{d}\mu\int \mathcal{D}\left(\frac{\rho(k)+\rho_h(k)}{\rho(k)}\right)e^{\mathcal{W}[\rho,\rho_h;\mu]} \end{aligned} \tag{2.111}$$

where the δ-function enforces particle density conservation, $e = \lim_{TH} E/L = \int \mathrm{d}k k^2\rho(k)$ is the energy density and C is a constant. Here, μ is the Lagrange multiplier associated to the density. The leading contribution to the partition function is found by extremizing the functional $\mathcal{W}[\rho,\rho_h;\mu]$,

$$\delta\mathcal{W} = L\int \mathrm{d}k\left\{-\beta(k^2-\mu) + \log\left(\frac{\rho_h(k)}{\rho(k)}\right) + \int\frac{\mathrm{d}\alpha}{2\pi}\varphi(k-\alpha)\log\left(1+\frac{\rho(\alpha)}{\rho_h(\alpha)}\right)\right\}\delta\rho(k) \tag{2.112}$$

[2]If we had worked with the bosonic formulation of the Bethe Ansatz the expression for the entropy would have been slightly different, since here the wave function has a fermionic nature: quantum numbers cannot be equal and thus a state is occupied or not.

This is easily obtained once we eliminate $\delta\rho_h$ by means of (2.107). Definig the *pseudoenergy* (synonimous of *dressed energy*) as,

$$\frac{\rho_h(k)}{\rho(k)} = \exp(\beta\epsilon(k)) \tag{2.113}$$

We find the famous *Yang-Yang equation*,

$$\epsilon(k) = \epsilon_0(k) - \beta^{-1}\int \frac{\mathrm{d}\alpha}{2\pi}\varphi(k-\alpha)\log\left(1+e^{-\beta\epsilon(\alpha)}\right) \tag{2.114}$$

where $\epsilon_(k) = k^2 - \mu$ is the *bare* energy. The integration measure in the partition function is,

$$\mathcal{D}\left(\frac{\rho-\rho_h}{\rho}\right) = \mathcal{D}(1+\epsilon) = \mathcal{D}(\epsilon) \tag{2.115}$$

so that the the change of variables (2.113) is consistent with the path integration. The name pseudo-energy comes from the following argument. First, the state k occupation number is,

$$n(k) = \frac{\rho(k)}{\rho_s(k)} = \frac{1}{1+e^{\beta\epsilon(k)}} \tag{2.116}$$

which has the same form of the *Fermi-Dirac* distribution for free fermions[3], with the pseudo-energy replacing the dispersion relation of free fermion. The entropy is,

$$\mathcal{S} = L\int\left[(2\pi)^{-1}\log\left(1+e^{-\beta\epsilon(k)}\right) + \beta^{-1}\epsilon_0\rho(k)\right]\mathrm{d}k \tag{2.117}$$

and consequently, the free energy is,

$$\mathcal{F} = \beta^{-1}\log\mathcal{Z} = E - \beta^{-1}\mathcal{S} = N\mu - \beta^{-1}L\int\frac{\mathrm{d}k}{2\pi}\log\left(1+e^{-\beta\epsilon(k)}\right) \tag{2.118}$$

This last equation shows that the physical interpretation of μ is that of chemical potential. The free energy has the same form of that of free fermions but with non trivial dispersion relation. We interpret the function ϵ as *dressed* energy. Due to excitations, the one dimensional system responds as whole, giving rise to a renormalization of bare single particle quantities. It is interesting, but not surprising [67], how the one dimensional system of bosons behaves like a dressed free fermionic system. Our last goal will be the computation of the one point functions and full counting statistics of particle density in the semi-classical limit of the LL model, namely of the classical field theory described by the NLSE and we will see that the fundamental quantity determining the steady state properties will be the root density. We will perform the semi-classical limit of the above equations later. For the moment the important thing to keep in mind is that solving (2.114) for the pseudo-energy allows to eliminate the hole density from (2.107) and to get the root density ρ: this is the fundamental quantity we are interested in. Since it encodes the information of the state of the system and it's fixed, once the temperature and the chemical potential have been chosen to a value, it is clear that its knowledge fully

[3]Indeed, we are using the fermionic formulation of the Bethe Ansatz.

determine the thermodynamics properties. All the information about the initial many-body system is contained in the S-matrix, thus, ultimately in the kernel φ appearing in all the integral equations. Once the kernel is given the thermodynamic description is fully determined after chosing an initial state. This state can be chosen at least in three different ways: the first, more accesible for an experimentalist, is to fix the temperature and the particle density; the second is to construct a particular root density; the third, also relevant for experiments, is to fix the *filling fraction* (aka quasi- particles occupation number). To see that the root density and the filling fraction contain the same information, we have to show that given one of them it is possible to get the other. The transformation from ρ to $n(k)$ it's trivial since we can solve (2.107) to get ρ_s and from (2.116) we get n. The transformation from n to ρ is non linear and reads,

$$n(k) = \frac{2\pi\rho(k)}{(p'(k))^{dr}} \tag{2.119}$$

where $p(k) = k$ and we have introduced the *dressing* operation and $(p(k)')^{dr}$ is the dressed momentum. Given a function $h(k)$, $(h(k))^{dr}$ satisfies the linear integral equation,

$$(h(k))^{dr} = h(k) + \int \frac{\mathrm{d}\alpha}{2\pi}\varphi(k-\alpha)n(\alpha)(h(\alpha))^{dr} \tag{2.120}$$

The proof is as follows. Define the linear operators,

$$\langle\lambda|T|\rho\rangle = \int \mathrm{d}\alpha\varphi(\lambda-\alpha)\rho(\alpha) \tag{2.121}$$

$$\langle\lambda|n|\rho\rangle = n(\lambda)\rho(\lambda) \tag{2.122}$$

Rewrite (2.107) and (2.116) as,

$$2\pi\,|\rho_s\rangle = |p_0'\rangle + T\,|\rho\rangle \tag{2.123}$$

$$|\rho\rangle = n\,|\rho_s\rangle \tag{2.124}$$

Thus,

$$2\pi\,|\rho_s\rangle = |p_0'\rangle + Tn\,|\rho_s\rangle \tag{2.125}$$

It follows thas,

$$2\pi(\mathbb{I} - (2\pi)^{-1}Tn)\,|\rho_s\rangle = |p_0'\rangle \tag{2.126}$$

and so,

$$2\pi\,|\rho_{Tss}\rangle = (\mathbb{I} - (2\pi)^{-1}Tn)^{-1}\,|p_0'\rangle \tag{2.127}$$

Finally,

$$2\pi\,|\rho\rangle = n(\mathbb{I} - (2\pi)^{-1}Tn)^{-1}\,|p_0'\rangle \tag{2.128}$$

Defining,

$$|p_0'\rangle^{dr} \equiv (\mathbb{I} - (2\pi)^{-1}Tn)^{-1}\,|p_0'\rangle \tag{2.129}$$

Multiplying both sides of by the inverse we find (2.120). We know that standard thermodynamics predicts equilibrium quantities using Gibbs Ensembles (GE). The TBA is indeed referred to the standard, *thermal*, GE. We know that integrable models exhibit an infinite set of conserved quantities, not only the density and the energy. The generalization of the TBA formalism which takes into account the GGE implied by integrability is somewhat called *Generalized Bethe Ansatz*. The generalization of (2.114) to many conserved quantities has been estabilished in Ref. [66] and reads,

$$\epsilon(k) = \Lambda(k, \underline{\mu}) - \int \frac{\mathrm{d}\alpha}{2\pi}\varphi(k-\alpha)\log\left(1+e^{-\epsilon(\alpha)}\right) \tag{2.130}$$

where this time the driving term is,

$$\Lambda(k, \underline{\mu}) = \sum_i \mu_i q_i(k) \tag{2.131}$$

and $q_i(k)$ is the conserved charge eigenvalues. The form of the eigenvalues are model-dependent but they are anyway easy expressions. For instance, in the LL model,

$$q_i(k) = k^i \tag{2.132}$$

and as easy to see, the expectation values of the charges are,

$$\langle \mathcal{Q}_n \rangle = \int \mathrm{d}k q_n(k)\rho(k) \tag{2.133}$$

In this case it is difficult to solve for the root density and the pseudoenergy due to the infinite terms present in Λ and the unknown Lagrange multipliers μ_i, which are in principle self-consistentely determined by,

$$\langle \mathcal{Q}_n \rangle = \frac{\partial \mathcal{F}}{\partial \mu_n} \tag{2.134}$$

where $\mathcal{F}$ is the free energy. Solving this system can be an insormountable problem, even for truncated GGEs. This is the main difficulty in solving the Generalized TBA equations. A step forward the solution to this problem has been made in Ref. [68], where the authors recognizes that the quantity to be sought to get knowledge of the initial state is the *root density*. Indeed, knowing this it is possible to find the total density of states through the linear integral equation (2.107) and then the pseudoenergy from (2.116). In the next Chapter we will introduce the Inverse Scattering Method, which will directly provide us we the identitfication of the root density in classical integrable field theories.

2.4.4 Semiclassical limit at TBA level

TBA equations for the LL model have the form,

$$\epsilon_q(\lambda) = \Lambda_q(\lambda, \{\underline{\beta}\}) - \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi_q(\lambda-\lambda')\log\left(1+e^{-\epsilon_q(\lambda')}\right)d\lambda' \tag{2.135}$$

where φ_q is the Lieb-Liniger kernel (2.87), ϵ_q the pseudo-energy and $\Lambda_q(\lambda, \{\underline{\beta}\}) = \sum_n \beta_n Q_n$ is the generalized Hamiltonian acting as driving term and the subscript indicates that they are all quantum quantities. Also, notice that we have absorbed the temperature in the definition of the pseudoenergy. In what follows we will encounter often very singular kernels, so that a prescription is needed to regularize them. We define,

$$\int \mathrm{d}\alpha\varphi(k-\alpha)f(\alpha) = -\mathcal{P}\int \mathrm{d}\alpha\zeta(k-\alpha)\frac{\mathrm{d}f(\alpha)}{\mathrm{d}\alpha} \tag{2.136}$$

where,

$$\frac{\mathrm{d}\zeta(k)}{\mathrm{d}k} = \varphi(k) \tag{2.137}$$

We start from the kernel, which in our conventions takes the form,

$$\varphi_q(\lambda) = \frac{4mg}{\lambda^2+(2mg)^2} \tag{2.138}$$

After rescaling as indicated in (1.185) we find,

$$\varphi_q(\lambda) \mapsto \varphi(\lambda) = \frac{4mg\hbar}{\lambda^2+(2mg\hbar)^2} \tag{2.139}$$

In the limit $\hbar \to 0$ the kernel is a δ function of its argument. If $\beta \mapsto \hbar\beta$ then $\Lambda_q \mapsto \hbar\Lambda$ and it remains to understand how the pseudo-energy scales. A non trivial equation is obtained if $\epsilon_q \to \log(\epsilon\hbar)$ [8]. We find at first order in $\hbar$,

$$\begin{aligned}\log(\hbar\epsilon(\lambda)) &\approx \hbar\Lambda(\lambda,\{\underline{\beta}\}) - \frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(\lambda-\lambda')\log\left(1+(\hbar\epsilon(\lambda'))^{-1}\right)d\lambda' \\ &\approx \hbar\Lambda(\lambda,\{\underline{\beta}\}) - \frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(\lambda-\lambda')\left[\hbar\epsilon(\lambda') - \log(\hbar\epsilon(\lambda'))\right]d\lambda'\end{aligned} \tag{2.140}$$

Note that,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\lambda\varphi(\lambda) = 1$$

so using the symmetry of the kernel we can write,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\lambda'\hbar\varphi(\lambda-\lambda')\epsilon(\lambda') = \hbar\Lambda(\lambda,\{\underline{\beta}\}) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\lambda'\varphi(\lambda-\lambda')\left[\log(\epsilon(\lambda')) - \log(\epsilon(\lambda))\right] \tag{2.141}$$

Simplfying $\hbar$ and taking $\hbar \to 0$ we find,

$$\epsilon(\lambda) = \Lambda(\lambda, \{\underline{\beta}\}) + \lim_{\hbar\to 0} \int_{-\infty}^{+\infty} \frac{d\lambda'}{2\pi} \frac{4mg}{(\lambda-\lambda')^2 + (2mg\hbar)^2} \left[\log(\epsilon(\lambda')) - \log(\epsilon(\lambda))\right] \tag{2.142}$$

where we have used that in this limit the kernel contributes a δ function Integrating by parts we get the final expression,

$$\epsilon(\lambda) = \Lambda(\lambda, \{\underline{\beta}\}) - 4mg\mathcal{P} \int_{-\infty}^{+\infty} \frac{d\lambda'}{2\pi} \frac{\partial_{\lambda'} \log \epsilon(\lambda')}{\lambda - \lambda'} \tag{2.143}$$

The semiclassical limit of TBA equations displays the same difficulties of the quantum case. The presence of the driving terms prevent us from finding easy the solutions. Now, suppose we knew the root density ρ. By a procedure analogous to the above it is easy to see that the semi-classical limit of the dressing operation becomes,

$$(h(k))^{dr} = h(k) - 4mg\mathcal{P} \int \frac{\mathrm{d}\alpha}{\lambda - \alpha} \partial_\alpha [n(\alpha)(h(\alpha))^{dr}] \tag{2.144}$$

We shall prove this formula because the kind of calculation is typical with respect what we will do later. Indicate quantum quantities with a subscript and classical one without any subscript. We have,

$$\frac{4mg\hbar}{\lambda^2 + (2mg\hbar)^2} = 2\pi\delta(\lambda) - 4mg\hbar\mathcal{P}\left(\frac{1}{2\pi\lambda}\partial_\lambda\right) + O(\hbar^2) \tag{2.145}$$

$$\frac{4mg\hbar^2}{\lambda^2 + (2mg\hbar)^2} = 2\hbar\pi\delta(\lambda) - 4mg\hbar^2\mathcal{P}\left(\frac{1}{2\pi\lambda}\partial_\lambda\right) + O(\hbar^3) \tag{2.146}$$

in distributional sense. From the definition of the filling,

$$n_q(\lambda) = \frac{1}{1 + e^{\epsilon(\lambda)}} \tag{2.147}$$

Again, $\epsilon_q \to \log(\hbar\epsilon)$ gives,

$$n_q(\lambda) = 1 - \hbar\epsilon(\lambda) + O(\hbar^2) \tag{2.148}$$

Thus, the classical dressing operation is obtained using these expressions in (2.120) and retaining terms of order $\hbar$,

$$(h_q(\lambda))^{dr} = h_q(\lambda) + (h(_q(\lambda))^{dr} - 4mg\hbar\mathcal{P} \int \frac{\mathrm{d}\alpha}{2\pi} \frac{1}{\lambda - \alpha} \partial_\alpha (h_q(\alpha))^{dr} - \hbar(h_q(\alpha))^{dr}\epsilon(\lambda) \tag{2.149}$$

Defining $(h_q(\alpha))^{dr} = \hbar^{-1} n(\alpha)(h(\alpha))^{dr}$ and using (2.151) we find the classical dressing operation above (notice that the $h_q(\lambda) \to h(\lambda)$, since charge eigenvalues remains unchanged going from quantum to classical). Note that in particular, due to (2.119), it follows that,

$$(p'(k))^{dr} = p'(k) - 4mg\mathcal{P} \int \mathrm{d}\alpha \frac{\partial_\alpha \rho(\alpha)}{k - \alpha} \tag{2.150}$$

Thus, if the root density is known we can use it to find the the filling fraction (2.119). This function can in turn be used in the classical version of the LeClaire-Mussardo series [37, 38] in combination with classical Form Factors and to give an expansion for expectation values of local observables. The route we want to take is different. Indeed in Ref. [69–72] it was found that also in classical integrable field theories the main object for the thermodynamics is the pseudoenergy, see Ref. [73] for a general framework. Generally, in calssical field theories, there are two distinct types of excitations: *radiative* and *solitonic* modes. In the NLSE model with $g > 0$, there are no solitonic modes. What is important for us is that to each mode is associated an occupation number function like (2.116) and like that we found in free models $\langle |A(k)|^2 \rangle$. Of course, like in the quantum case, it is related to the pseudoenergy. For radiative modes, their relation reads,

$$n(k) = \frac{1}{\epsilon(k)} \tag{2.151}$$

Withouth going into the details of the derivation, the expression can be justified in two different ways. The first is that in free theories this is trivially true, as we have seen when we discussed free models. The second it to perform the semiclassical limit of the free energy (2.118) and using the prescription (2.136). We find,

$$\mathcal{F} = N\mu - L \int \frac{\mathrm{d}k}{2\pi} \log(\epsilon(k)) \tag{2.152}$$

Comparing to (2.130) or (2.135) we see that,

$$n(k) = -\frac{\partial}{\partial \epsilon(k)} - \log(\epsilon(k)) = \frac{1}{\epsilon(k)} \tag{2.153}$$

where it is understood the $\epsilon(k)$ must satisfy generalized TBA equations. This means that knowing the root density we can find the pseudoenergy via (2.151). It is the only missing ingredient and in the next Chapter we will se how to compute it.

2.4.5 UV finitness

It is well known that UV divergences are present not only in free theories: electromagnetism is a simple example where thermal averages fail to give finite result. The same problem appears in other theories like the NLS model. The reasong in that for large momenta the field can be very "rough" due to the presence of radiative modes. Indeed, recalling the expression of the filling $n(k) = 1/\epsilon(k)$ it is related to the root density by $2\pi\rho(k) = (p'(k))^{dr} n(k)$. At large momenta, the kernel appearing in TBA equations as well in the dressing operation goes to zero thus we have $2\pi\rho(k) \sim n(k) = 1/\epsilon(k)$. For thermal states $\epsilon(k) \sim k^2$ so that recalling the expression of for the expectation of a charge,

$$\langle \mathcal{Q}_n \rangle = \int \rho(k) k^n \mathrm{d}k \sim \int k^{n-2} \mathrm{d}k \tag{2.154}$$

we see that even the energy is not finite on thermal states. The same happens for other GGEs involving arbitrary number of local charges. The reasoning is the same as the above since for large momenta the main contribution to the pseudoenergy comes from the driving term in generalized TBA equations (2.143). In real space this is due to the

fact a GGE involving local charges up to a certain "spin" does not regularize the field in order to render finite high order derivatives. By the same argument we recognize why in discussing free models we called $|A(k)|^2$ filling fraction.

Chapter 3

Inverse Scattering Method

In the previous Section we have seen how quantum theories correspond to classical ones in an appropraite semiclassical limit. We have obtained classical generalized TBA equations but we sill have the problem to determine the root density. In this respect we illustrate the Inverse Scattering Method, which will utimately provide us with the classical version of the root density, appearing in (2.107). The inverse scattering method (IST) is a powerful tool in mathematical physics used to solve a particular class of non linear PDE's. It has been used to find solution for the Korteweg-deVries (KdV) equation in the theory of shallow water, the Sine-Gordon (SG) equation, the Non Linear Schrödinger equation (NLSE) and many others. Originally it was developed by Kruskal, Gardner, Greene and Miura[34]. They noted that if $\psi(x,t)$ is a solution of the KdV equation then it can be viewed as a the potential associated to a certain scattering problem in quantum mechanics. Here we want to give an overview of the main ingredients of the method. Standard referencens are [32, 33, 74]. Focusing on one dimensional systems, an evolution equation has the form,

$$L(t)\psi = f(\psi, \psi_x, \psi_{xx}, \psi_{xxx}, \dots) \tag{3.1}$$

where the function f can be more or less arbitrary and L is a linear operator acting as time derivative. For example,

- (Kdv) $L(t) = \partial_t \quad f = 6\psi\psi_x - \psi_{xxx}$
- (SG) $L(t) = \partial_t^2 \quad f = \psi_{xx} - sin(\psi)$
- (NLSE) $L(t) = i\partial_t \quad f = -\psi_{xx} + 2g|\psi|^2\psi$

As can be seen these equations are non linear and so standard method from linear algebra and Fourier analysis cannot be applied. What is common to the equations is that they display wave solutions: these were observed for the first time by Scott Russel when studying efficient design for canal boats and by Zabusky and Kruskal using computer simulations. For an hisorical review see Ref. [75]. In the words of Russel,

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its

> height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation

Anyway, the IST is a method to efficiently find solutions to difficult problems and there are three main steps in its application:

1. Solve the associated scattering problem at time 0, i.e. find the *scattering data* S. In practice, S is the traditional scattering matrix in physics. Its elements play a fundamental role in the determinantion of the solutions. This is the direct problem.
2. Evolve the scattering data according to the evolution equation, that is to $L(t)$.
3. Solve the inverse problem: from the solution of the scattering problem recover the unknown potential at time t.

Schematically we can say:

$$\psi(x,0) \xrightarrow{1} S(0) \xrightarrow{2} S(t) \xrightarrow{3} \psi(x,t)$$

3.1 Non-Linear Schrödinger model

3.1.1 Hamiltonian, Lagrangian and Phase Space

The NLSE is a Hamiltonian system described by the partial differential equation for the complex field $\psi(x,t) : A \subset \mathbb{R} \times [0,+\infty]$,

$$i\partial_t \psi(x,t) = -\partial_x^2 \psi(x,t) + 2g|\psi(x,t)|^2 \psi(x,t) \tag{3.2}$$

The Hamiltonian is,

$$H = \int \mathrm{d}x \left\{ |\partial_x \psi(x)|^2 + g|\psi(x)|^4 \right\} \tag{3.3}$$

Appropriate boundary conditions (BC) must be given in order to specify the problem. In principle we consider two kinds of BC,

- Open boundary conditions: the field is defined on the real line $(-\infty,+\infty)$ and belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$ that is $|x^\alpha \partial_x^\beta \psi(x)| \to 0$ as $x \to \infty$ for every $\alpha \in \mathbb{R}$,$\beta \in \mathbb{N}$.
- Periodic boundary conditions: the field is defined on the segment $[0,L]$ and it and all its derivatives are periodic on this domain e.g. $\partial_x^\beta \psi(x,t)|_{x=0} = \partial_x^\beta \psi(x,t)|_{x=L}$ for every $t \geq 0$ and for every $\beta \in \mathbb{N}$

BC and NLSE define the NLS model. These two types of boundary conditions, after the thermodynamic limit is considered, implement the physical situation of zero energy density and finite energy density as we will discuss below. First we fix notations for

what comes later following[32]. Given the field ψ the algebra of observables we consider is spanned by elements of the following type:

$$F(\psi,\bar{\psi}) = c + \sum_{\substack{n,m=0 \\ (n,m)\neq(0,0)}}^{+\infty} \int \prod_{i=1}^{n} dy_i \prod_{j=1}^{m} dz_j c_{nm}(\{y\},\{z\}) \prod_{i=1}^{n} \bar{\psi}(y_i) \prod_{j=1}^{m} \psi(z_j) \tag{3.4}$$

where $c_{nm}(\{y\},\{z\}) \doteq c(y_1,\ldots,y_n,z_1,\ldots,z_m)$ are tempered distributions symmetric separately in y_i and z_i satisfying the reality condition:

$$c_{nm}(\{y\},\{z\}) = \bar{c}_{nm}(\{z\},\{y\}) \tag{3.5}$$

Tempered distributions are continuous functionals on the Schwartz class $\mathcal{S}(\mathbb{R})$. The above condition is necessary to guarantee the reality of observables. The variational derivative of a functional is given by,

$$\delta F(\psi) = \int dx \frac{\delta F}{\delta \psi(x)} \delta\psi(x) \tag{3.6}$$

This definition naturally extends by linearity to more complicated functionals. The algebra product between $F = F(\psi,\bar{\psi})$ and $G = G(\psi,\bar{\psi})$ is defined by the Poisson bracket,

$$\{F,G\} = i \int dx \left[\frac{\delta F}{\delta\psi(x)} \frac{\delta G}{\delta\bar{\psi}(x)} - \frac{\delta F}{\delta\bar{\psi}(x)} \frac{\delta G}{\delta\psi(x)} \right] \tag{3.7}$$

Note the presence of the imaginary unit, needed to be consistent with antisymmetry of Poisson brackets and complex conjugation. In particular we find the basic relations,

$$\{\psi(x),\bar{\psi}(y)\} = i\delta(x-y) \tag{3.8}$$

$$\{\psi(x),\psi(y)\} = \{\bar{\psi}(x),\bar{\psi}(y)\} = 0 \tag{3.9}$$

and more importantly,

$$\frac{\delta F}{\delta\psi(x)} = -i\{F,\bar{\psi}(x)\} \quad \frac{\delta F}{\delta\bar{\psi}(x)} = i\{F,\psi(x)\} \tag{3.10}$$

We know that each observable generates a one parameter group of transformations. The hamiltonian generates time translations so that:

$$\partial_t\psi(x,t) = \{H,\psi(x,t)\} = -i\frac{\delta H}{\delta\bar{\psi}(x)} \tag{3.11}$$

$$\partial_t\bar{\psi}(x,t) = \{H,\bar{\psi}(x,t)\} = i\frac{\delta H}{\delta\psi(x)} \tag{3.12}$$

These are the Hamilton's equation of motion and give the dynamics of the field ψ. Equation (3.2) is easily derived from the Hamiltonian (3.3) and Hamilton's equations (3.12). The above applies in the case of open boundary conditions with the field going to zero at infinity. In case of periodic boundary conditions we take $\psi(L) = \psi(0)$ and send

$L \to +\infty$. In fact, being the hamiltonian an integral of motion we have,

$$0 \leq \lim_{L\to\infty} \frac{|E|}{L} = \lim_{L\to\infty} \frac{1}{L} \left| \int_0^L dx \mathcal{H}(\psi(x), \partial_x \psi(x)) \right| \leq \lim_{L\to\infty} \sup_{x\in[0,L]} |\mathcal{H}(\psi(x), \partial_x \psi(x))| \leq \infty \tag{3.13}$$

Physical field configurations must satisfy the above inequality. For example $\psi(x) = \sin(\frac{2\pi}{L}x)$ trivially satisfy the bound and is indeed a physical configuration. Sticking for the moment to the classical realm and remanding to the quantization in the next paragraph we can give an idea of why the NLSE is relevant in physical applications [6, 33, 76]. A wide class of physical phenomena are described by the equation,

$$\partial_t^2 \psi(x,t) = v_0 \partial_x^2 \psi(x,t) \tag{3.14}$$

which describes the propagation a persistent wave travelling with constant velocity v_0. In deriving this equation, usually, we do three different approximantions: fisrt, we assume there is no dissipation, thus we ask for $t \to -t$ symmetry; second, the amplitudes of oscillations are sufficiently small so that we can ignore non linear corrections; third, we assume there is no dispersion in the wave length range under consideration. Relaxing one of these hypotesis leads to reject the validity of the linear wave equation. Clearly, in real systems, dispersion, dissipation and non linearity are by no means negligible. Interestingly, relaxing a single assumption leads to evolution equations which still are tractable from a mathematical point of view and still catch non trivial physical effects. For instance, consider oscillations in a plasma. These electron oscillations take place at the plasma frequency $\epsilon_0 = \sqrt{\frac{4\pi n_e}{m}}$, where n_e is the electron density and m the mass. Heavy ions do not partecipate to these oscillations. Charge neutrality requires electrons and ions density to be equal and both cannot have a variable component of frequency of order $O(\epsilon_0)$. Neverthless, time averaged force acting on the plasma can generate oscillations in the total density. If the field is of the type,

$$\psi = \psi_0 e^{i(k_0 x - \epsilon t)} \tag{3.15}$$

with ψ_0 slowly varying in space and time, then the spectral expansion of the field would be peaked aroung k_0. This means that the dispersion relation can be expanded in powers of $k - k_0$,

$$\epsilon(k) = \epsilon_0 + v_0(k - k_0) + \beta(k - k_0)^2 \tag{3.16}$$

This dispersion relation correspond to the equation,

$$i\partial_t \psi = \epsilon_0 \psi + v_0 \left(\frac{1}{i}\partial_x - k_0 \right) \psi + \beta \left(\frac{1}{i}\partial_x - k_0 \right)^2 \psi \tag{3.17}$$

Inserting (3.15) we obtain the equation,

$$i\left(\partial_t \psi_o + v_0 \partial_x \psi_0\right) = -\beta \partial_x^2 \psi_0 \tag{3.18}$$

The weak non linearity is taken into account by letting,

$$\epsilon(k_0) = \epsilon_0 + |\psi_0|^2 \tag{3.19}$$

thus obtaining an NLSE-like wave equation.

3.2 Scattering Problem and Zero Curvature Condition

The remarkable observation which led to the development of the inverse scattering trasform is that the evolution equation can be seen as a compatibility condition for the scattering problem defined by the following system,

$$\partial_x F(x,t) = U_\lambda(x,t)F(x,t) \tag{3.20a}$$
$$\partial_t F(x,t) = V_\lambda(x,t)F(x,t) \tag{3.20b}$$

where $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and U_λ, V_λ are matrices depending parametrically by the spectral parameter and by the field. Assuming sufficient regularity of the matrices U and V the equations are compatible only if the so called zero curvature condition holds,

$$\partial_t U_\lambda - \partial_x V_\lambda + [U_\lambda, V_\lambda] = 0 \tag{3.21}$$

The terminology comes from Riemaniann geometry and the connection with it is actually more than that. In fact given a vector bundle one can define a connection Γ and use it to implement parallel transport of tensors. In index notation the covariant derivative is,

$$D_\mu S^\rho = \partial_\mu S^\rho + \Gamma^\rho_{\mu\nu} S^\nu$$

The Rieman curvature tensor quantifies the non commutativity of covariant derivatves,

$$[D_\mu, D_\nu] S^\rho = R^\rho_{\sigma\mu\nu} S^\sigma$$

and by explicit computation one can check,

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\tau}\Gamma^\tau_{\nu\sigma} - \Gamma^\rho_{\nu\tau}\Gamma^\tau_{\mu\sigma} \tag{3.22}$$

It is now very clear the connection between (3.22) and (3.21) once one interprets U and V as connection coefficients. To see that this connection can be seriously exploited we define the covariant derivative as,

$$D_0 = \partial_0 - A_0 \tag{3.23}$$
$$D_1 = \partial_1 - A_1 \tag{3.24}$$

with the identification $\partial_0 = \partial_x$, $\partial_1 = \partial_t$, $A_0 = U$, $A_1 = V$. Equation (3.21) reads,

$$[D_0, D_1] = 0 \tag{3.25}$$

Consider the differential form $A = A_\mu dx^\mu = U_\lambda dx + V_\lambda dt$ and a closed path γ in space time. We have $DA = 0$ and then application of Stokes theorem gives,

$$\mathcal{P}\exp\left(\oint_\gamma A_\mu dx^\mu\right) = \mathcal{P}\exp\left(\int_\gamma DA\right) = \mathbb{I} \tag{3.26}$$

where $\mathcal{P}$ is the path ordering operator. The above operator implements the parallel transport. We now consider the closed path (x,t_1), (y,t_1), (y,t_2), (x,t_2) and (x,t_1) and define the propagators (monodromy matrices),

$$T_\lambda(x,y;t) = \mathcal{P}\exp\left(\int_x^y U_\lambda(x',t)dx'\right) \quad S_\lambda(t_1,t_2;x) = \mathcal{P}\exp\left(\int_{t_1}^{t_2} V_\lambda(x,t')dt'\right) \tag{3.27}$$

From these definition it follows that these matrices satisfy,

$$\partial_x T_\lambda(x,y;t) = T_\lambda(x,y,;t)U_\lambda(x,t) \quad \partial_y T_\lambda(x,y;t) = U_\lambda(y,t)T_\lambda(x,y,;t) \tag{3.28}$$

and an analogous pair for S_λ. The importance of the propagators is that they can be used to "evolve" the solution in space time. By direct substitution one can check that for example,

$$F(y,t) = T_\lambda(x,y,t)F(x,t) \quad y < x \tag{3.29}$$

Using (3.26) and (3.27) we can write,

$$S_\lambda(t_2,t_1;x)T_\lambda(y,x;t_2)S_\lambda(t_1,t_2;y)T_\lambda(x,y;t_1) = \mathbb{I} \tag{3.30}$$

From the elementary properties $S_\lambda^{-1}(t_1,t_2;x) = S_\lambda(t_2,t_1;x)$ and $T_\lambda^{-1}(x,y;t) = T_\lambda(y,x;t)$ we find,

$$T_\lambda(x,y;t_1) = S_\lambda^{-1}(t_1,t_2;y)T_\lambda(x,y;t_2)S_\lambda(t_1,t_2;x) \tag{3.31}$$

The point of this derivation is that if we find two points in space-time such that $V_\lambda(x,t) = V_\lambda(y,t)$ the above relation says that,

$$F_\lambda(x,y) = Tr(T_\lambda(x,y;t)) \tag{3.32}$$

is time independent because $T_\lambda(x,y;t)$ and $M(\lambda) = V_\lambda(x,t)$ form a so called Lax pair,

$$\dot{T_\lambda} = [M, T_\lambda] \tag{3.33}$$

Then for any value of the spectral parameter the quantity defined in eq (3.32) can be used to generate an infinite set of conserved quantities. The last result of this section is the proof of (3.33), the equation of motion for the monodromy matrix (from this moment we refer T_λ with this name). Setting $t_1 = t$, $t_2 = t + \delta t$,

$$T_\lambda(x,y;t+\delta t) = T_\lambda(x,y;t) + \delta t\frac{dT_\lambda(x,y,t)}{dt}O(\delta t^2) \tag{3.34}$$

$$S_\lambda(t,t+\delta t;x) = \mathcal{P}\exp\left(\int_t^{t+\delta t} V_\lambda(x,t')dt'\right) = \mathcal{P}\exp\left(\delta t V_\lambda(x,t)\right) \tag{3.35}$$

$$= \mathbb{I} + \delta t V_\lambda(x,t) + O(\delta t^2) \tag{3.36}$$

and inserting in (3.31) we find the evolution equation for the transfer matrix:

$$\frac{dT_\lambda(x,y;t)}{dt} = V_\lambda(y,t)T_\lambda(x,y;t) - T_\lambda(x,y;t)V_\lambda(x,t) \tag{3.37}$$

3.3 The NLSE on the Whole Line: Rapidly Decrasing Case

3.3.1 Transfer Matrix

Here we apply the theory of the previous section to the classical NLS model in the rapidly decrasing case in the repulsive regime whene it is known that the spectrum is purely continuous and no bound state exists. This has the implication that since solitons show up from the discrete spectrum of a scattering problem they will not exist in this regime. Preliminarly we recall the definition of Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.38}$$

$$\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2) \tag{3.39}$$

and their (anti-)commutation relations,

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij} \tag{3.40}$$

The scattering problem we consider is (3.20) with the identifications,

$$U_\lambda = U_0 + \lambda U_1 \tag{3.41}$$

$$V_\lambda = V_0 + \lambda V_1 + \lambda^2 V_2 \tag{3.42}$$

$$U_0 = \sqrt{g}(\bar\psi\sigma_+ + \psi\sigma_-) = \sqrt{g}\begin{pmatrix} 0 & \bar\psi \\ \psi & 0 \end{pmatrix} \tag{3.43}$$

$$U_1 = \frac{1}{2i}\sigma_3 = \frac{1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.44}$$

$$V_0 = ig|\psi|^2\sigma_3 - i\sqrt{g}(\bar\psi_x\sigma_+ - \psi_x\sigma_-) = i\sqrt{g}\begin{pmatrix} \sqrt{g}|\psi|^2 & -\bar\psi_x \\ \psi_x & -\sqrt{g}|\psi|^2 \end{pmatrix} \tag{3.45}$$

$$V_1 = -U_0 \tag{3.46}$$

$$V_2 = -U_1 \tag{3.47}$$

Note that here we abbreviate $\partial_x\psi = \psi_x$ and $\partial_t\psi = \psi_t$ and the bar means complex conjugation. The first remark is the *involution property*,

$$\sigma_1 U_\lambda \sigma_1 = U_{\bar\lambda} \quad \forall x,t \tag{3.48}$$

which follows from the properties of Pauli matrices and the definition o U_λ (for $g < 0$ use σ_2 instead of σ_1). As a consequence the propagator,

$$\sigma_1 T_\lambda(x,y;t)\sigma_1 = \bar T_{\bar\lambda}(x,y;t) \tag{3.49}$$

Then if,

$$\sigma_1 T_\lambda(x,y;t)\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_\lambda(x,y;t) & \beta_\lambda(x,y;t) \\ \gamma_\lambda(x,y;t) & \delta_\lambda(x,y;t) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \delta_\lambda(x,y;t) & \gamma_\lambda(x,y;t) \\ \beta_\lambda(x,y;t) & \alpha_\lambda(x,y;t) \end{pmatrix} \tag{3.50}$$

Because of (3.49),

$$T_\lambda(x,y;t) = \begin{pmatrix} \alpha_\lambda(x,y;t) & \beta_\lambda(x,y;t) \\ \bar{\beta}_{\bar{\lambda}}(x,y;t) & \bar{\alpha}_{\bar{\lambda}}(x,y;t) \end{pmatrix} \tag{3.51}$$

Since $TrU_\lambda = 0$ we have also,

$$det\left(T_\lambda(x,y,t)\right) = 1 \quad \forall \lambda \in \mathbb{C} \quad \forall x,t \in \mathbb{R} \tag{3.52}$$

$$|\alpha_\lambda(x,t)|^2 - |\beta_\lambda(x,t)|^2 = 1 \quad \forall \lambda, x, t \in \mathbb{R} \tag{3.53}$$

By (3.32) the trace is actually time independent,

$$\partial_t Tr T_\lambda(x,y;t) = 0 \tag{3.54}$$

(3.20a) in the limit $x \to +\infty$ becomes,

$$\partial_x F(x,t) = \frac{\lambda}{2i}\sigma_3 F(x,t) \tag{3.55}$$

and there are two *linearly independent* solutions,

$$\psi^+_\lambda(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\frac{1}{2}i\lambda x} \tag{3.56}$$

$$\psi^-_\lambda(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\frac{1}{2}i\lambda x} \tag{3.57}$$

or in matrix notation,

$$\Psi_\lambda(x) = \begin{pmatrix} \psi^+_\lambda(x) & \psi^-_\lambda(x) \end{pmatrix} = \begin{pmatrix} e^{-\frac{1}{2}i\lambda x} & 0 \\ 0 & e^{\frac{1}{2}i\lambda x} \end{pmatrix} \tag{3.58}$$

A theorem from scattering theory [77], assures the existence of the so called *Jost solutions* (for rapidly decreasing potentials), defined by their asymptotic behavior,

$$\mathcal{F}_\lambda = \begin{pmatrix} f^+_\lambda & f^-_\lambda \end{pmatrix} \to \Psi_\lambda \quad x \to +\infty \tag{3.59}$$

$$\mathcal{G}_\lambda = \begin{pmatrix} g^+_\lambda & g^-_\lambda \end{pmatrix} \to \Psi_\lambda \quad x \to -\infty \tag{3.60}$$

These resemble the solutions of a scattering problem for the Schrödinger equation in quantum mechanics where the boundary conditions are plane waves. By virtue of (3.29) it holds,

$$\mathcal{F}_\lambda(x,t) = \lim_{x_0 \to +\infty} T_\lambda(x_0,x,t)\Psi_\lambda(x) \tag{3.61}$$

$$\mathcal{G}_\lambda(x,t) = \lim_{x_0 \to -\infty} T_\lambda(x_0,x,t)\Psi_\lambda(x) \tag{3.62}$$

The involution property (3.48) reflects on the free and Jost solutions. Since $\bar{\psi}^+_\lambda = \sigma_1 \psi^-_\lambda$ for real λ,

$$\sigma_1 \Psi_\lambda \sigma_1 = \bar{\Psi}_\lambda \tag{3.63}$$
$$\sigma_1 \mathcal{F}_\lambda \sigma_1 = \bar{\mathcal{F}}_\lambda \quad \forall \lambda \in \mathbb{R} \tag{3.64}$$
$$\sigma_1 \mathcal{G}_\lambda \sigma_1 = \bar{\mathcal{G}}_\lambda \tag{3.65}$$

Now we define the *transfer matrix* as the matrix connecting the two basis of solutions at $\pm\infty$,

$$\mathcal{F}_\lambda(x,t) = \mathcal{G}_\lambda(x,t)\mathbb{T}_\lambda(t) \tag{3.66}$$

From this we find,

$$\mathbb{T}_\lambda(t) = \mathcal{G}^{-1}_\lambda(x,t)\mathcal{F}_\lambda(x,t) = \lim_{\substack{x_0\to-\infty \\ x_1\to+\infty}} \Psi^{-1}_\lambda(x_0) T_\lambda(x_1,x_0,t)\Psi_\lambda(x_1) \tag{3.67}$$

where we have used (3.29) multiple times. The transfer matrix for real λ has the same structure and properties of the propagator,

$$\mathbb{T}_\lambda(t) = \begin{pmatrix} a_\lambda(t) & \bar{b}_\lambda(t) \\ b_\lambda(t) & \bar{a}_\lambda(t) \end{pmatrix} \tag{3.68}$$
$$|a_\lambda(t)|^2 - |b_\lambda(t)|^2 = 1 \quad \forall \lambda, x, t \in \mathbb{R} \tag{3.69}$$

The matrix,

$$\Gamma_\lambda(x,t) = \lim_{x_0\to\infty} T_\lambda(x_0,x,t) \tag{3.70}$$

can be shown to satisfy a very useful integral equation. From (3.61) it follows that,

$$\Gamma_\lambda(x,t) = \Psi^{-1}_\lambda \mathcal{F}_\lambda(x,t) \tag{3.71}$$

Differentiating,

$$\begin{aligned} \partial_x \Gamma_\lambda &= \left(\partial_x \Psi^{-1}_\lambda\right) \mathcal{F}_\lambda + \Psi^{-1}_\lambda \left(\partial_x \mathcal{F}_\lambda\right) \\ &= -\Psi^{-1}_\lambda \left(\partial_x \Psi_\lambda\right) \Psi^{-1}_\lambda \mathcal{F}_\lambda + \Psi^{-1}_\lambda U_\lambda \mathcal{F}_\lambda \\ &= -\Psi^{-1}_\lambda \left(\partial_x \Psi_\lambda\right) \Gamma_\lambda + \Psi^{-1}_\lambda U_\lambda \Psi_\lambda \Gamma_\lambda \end{aligned} \tag{3.72}$$

that means,

$$\partial_x \Gamma_\lambda = \tilde{M}_\lambda \Gamma_\lambda \tag{3.73}$$
$$\tilde{M}_\lambda = \Psi^{-1}_\lambda \left(U_\lambda \Psi_\lambda - \partial_x \Psi_\lambda\right) = \begin{pmatrix} 0 & \sqrt{g}\bar{\psi} e^{i\lambda x} \\ \sqrt{g}\psi e^{-i\lambda x} & 0 \end{pmatrix} \tag{3.74}$$

This is solved by a space-ordered exponential or by iteration,

$$\Gamma_\lambda(x,t) = \mathbb{I} + \int_{-\infty}^{x} dy \tilde{M}_\lambda(y,t)\Gamma_\lambda(y,t) \tag{3.75}$$

This integral representation is very useful since it can be used to generate a perturbative expansion of the coefficients in the*transfer matrix* (3.68). Indeed,

$$\Gamma_\lambda(x,t) = \mathbb{I} + \int_{-\infty}^{x} \tilde{M}_\lambda(y,t) + \int_{-\infty}^{x} dy \int_{-\infty}^{y} dz \tilde{M}_\lambda(y,t)\tilde{M}_\lambda(z,t) + \dots \tag{3.76}$$

which, taking the limit $x \to +\infty$ gives,

$$a(\lambda) = 1 + g\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \theta(y-z)\bar{\psi}(y,t)e^{i\lambda y}\psi(z,t)e^{-i\lambda z} + \dots \tag{3.77}$$

Introducing the Fourier Transform of the step function,

$$\tilde{\theta}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\theta(x)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}\left(\pi\delta(k) - i\mathcal{P}\left(\frac{1}{k}\right)\right) \tag{3.78}$$

we find,

$$\begin{aligned} a(\lambda) &= 1 + g\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{\theta}(k)e^{iy(k+\lambda)}\bar{\psi}(y)e^{-iz(\lambda+k)}\psi(z) \\ &= 1 + g\sqrt{2\pi}\int_{-\infty}^{+\infty} dk\tilde{\theta}(k)\tilde{\bar{\psi}}(-k-\lambda)\tilde{\psi}(\lambda+k) \\ &= 1 + g\pi\tilde{\bar{\psi}}(-\lambda)\tilde{\psi}(\lambda) - gi\mathcal{P}\int_{-\infty}^{+\infty} dk \frac{\tilde{\bar{\psi}}(-k-\lambda)\tilde{\psi}(\lambda+k)}{k} \\ &= 1 + g\pi|\tilde{\psi}(\lambda)|^2 - gi\mathcal{P}\int_{-\infty}^{+\infty} dk \frac{|\tilde{\psi}(k)|^2}{k-\lambda} \end{aligned} \tag{3.79}$$

***Remark* 1.** *We have used the fact that,*

$$\tilde{\bar{\psi}}(k) = \tilde{\bar{\psi}}(-k) \tag{3.80}$$

In the following we will use this expression to see how the action-variable formulation of the integrable theory is connected to the one in the free thery.

3.3.2 Conserved Charges and Local Densities: NLSE

The derivation of conserved charges and local densities exploits a series expansion of the transfer matrix coefficient a_λ where the coefficients can be recursively determined. We make the Ansatz [9],

$$T_\lambda(x,y;t) = (\mathbb{I} + W(x))\, e^{Z(x)}C(y) \tag{3.81}$$

where we suppress for the moment the time and λ dependence, $C(y)$ is fixed by the boundary condition $T_\lambda(x,x) = \mathbb{I}$ and Z is diagonal while W is off-diagonal. Substituting in (3.28),

$$W'(x) + W(x)Z'(x) = U^{(O)}(x) + U^{(D)}W(x) \tag{3.82}$$

$$Z'(x) = U^{(D)}(x) + U^{(O)}(x)W(x) \tag{3.83}$$

where we have used that the product a diagonal matrix and an off diagonal one is off diagonal and the product of two off-diagonal matrices is diagonal. After eliminating $Z'(x)$ from the first equation,

$$W'(x) + [W(x), U^{(D)}(x)] + W(x)U^{(O)}W(x) = U^{(O)}(x) \tag{3.84}$$

For definitness take,

$$W(x) = \begin{pmatrix} 0 & -w_-(x) \\ w_+(x) & 0 \end{pmatrix} \quad Z(x) = \begin{pmatrix} z_+(x) & 0 \\ 0 & z_-(x) \end{pmatrix} \tag{3.85}$$

subsitute back in (3.84) and get,

$$\sqrt{g}\psi w_-^2(x) = w_-^{'}(x) + i\lambda w_-(x) + \sqrt{g}\bar{\psi} \tag{3.86a}$$
$$\sqrt{g}\bar{\psi} w_+^2(x) = -w_+^{'}(x) + i\lambda w_+(x) + \sqrt{g}\psi \tag{3.86b}$$

These are Riccati equations and an explicit solution is difficult to find. What one usually does is to try a power series expansion. In all the above we suppressed the parametric dependence on the spectral parameter λ. Matrices Z and W depend on it. We try a series solution as,

$$w_\pm(x) = \sum_{n=1}^{+\infty} \lambda^{-n} w_{\pm,n}(x) \tag{3.87}$$

Equating powers of λ we find the initial conditions,

$$w_{-,1} = i\sqrt{g}\bar{\psi} \tag{3.88a}$$
$$w_{+,1} = i\sqrt{g}\psi \tag{3.88b}$$

and using the Cauchy product formula,

$$\sum_{n=1}^{+\infty} a_n \sum_{m=1}^{+\infty} b_m = \sum_{n=1}^{+\infty} c_n \tag{3.89}$$
$$c_n = \sum_{k=1}^{n} a_n b_{n-k+1} \tag{3.90}$$

we get a recursion equation determining all the coefficients,

$$\sqrt{g}\psi \sum_{n=1}^{+\infty} \lambda^{-n-1} \sum_{k=1}^{n} w_{-,k} w_{-,n-k+1} = \sum_{n=1}^{+\infty} \lambda^{-n} w'_{-,n} + i \sum_{n=1}^{+\infty} \lambda^{-n+1} w_{-,n} + \sqrt{g}\bar{\psi} \tag{3.91}$$

and so,

$$i w_{-,n+1} = -w'_{-,n} + \sqrt{g}\psi \sum_{k=1}^{n-1} w_{-,k} w_{-,n-k} \tag{3.92}$$

In the same way we find,

$$iw_{+,n+1} = w'_{+,n} + \sqrt{g}\bar{\psi}\sum_{k=1}^{n-1} w_{+,k}w_{+,n-k} \tag{3.93}$$

Inserting this result in the equation for Z we find,

$$z'_{+}(x) = \frac{\lambda}{2i} + \sqrt{g}\bar{\psi}\sum_{n=1}^{+\infty}\lambda^{-n}w_{+,n} \tag{3.94a}$$

$$z'_{-}(x) = -\frac{\lambda}{2i} - \sqrt{g}\psi\sum_{n=1}^{+\infty}\lambda^{-n}w_{-,n} \tag{3.94b}$$

Since the coefficients of these expansions are polynomials in ψ and $\bar{\psi}$ and ψ,$\bar{\psi} \to 0$ as $x \to \infty$ it follows that $w_{\pm,k} \to 0$ also, in this limit. From equation (3.67),

$$\delta_\lambda = Tr\mathbb{T}(\lambda) = Tr\lim_{\substack{x_0\to-\infty\\x_1\to+\infty}}\Psi_\lambda^{-1}(x_0)T_\lambda(x_1,x_0,t)\Psi_\lambda(x_1) \tag{3.95}$$

$$= \lim_{\substack{x_0\to-\infty\\x_1\to+\infty}} TrT_\lambda(x_1,x_0,t) \tag{3.96}$$

$$= 2\cos(\phi_\lambda) \tag{3.97}$$

In the above calculation we have used the continuity and the ciclicity of the trace and the unimodularity of the transfer matrix. This last property allows us to define,

$$\mathbb{T}_\lambda = \begin{pmatrix} e^{i\phi_\lambda} & 0 \\ 0 & e^{-i\phi_\lambda}\end{pmatrix} \tag{3.98}$$

Comparing with (3.68) we find,

$$\log a(\lambda) = i\phi_\lambda = \sum_{n=1}^{+\infty}\lambda^{-n}I_n \tag{3.99}$$

with,

$$I_n := \sqrt{g}\int_{-\infty}^{+\infty} dx\bar{\psi}(x)w_{+,n}(x) \tag{3.100}$$

Note that because of the initial conditions (3.88a)-(3.88b) and the recursion formulas (3.92)-(3.93) a factor $\sqrt{g}$ can always be factored out. Also a factor $\sqrt{g}$ is present in the definition of the conserved charges I_n. Thus, we can write the expansion (3.99) as,

$$\log a(\lambda) = i\phi_\lambda = g\sum_{n=1}^{+\infty}\lambda^{-n}I_n \tag{3.101}$$

with,

$$I_n := \int_{-\infty}^{+\infty} dx\bar{\psi}(x)w_{+,n}(x) \tag{3.102}$$

and,

$$iw_{-,n+1} = -w'_{-,n} + g\psi \sum_{k=1}^{n-1} w_{-,k} w_{-,n-k} \tag{3.103}$$

$$iw_{+,n+1} = w'_{+,n} + g\bar{\psi} \sum_{k=1}^{n-1} w_{+,k} w_{+,n-k} \tag{3.104}$$

For what we said in the prevoius section, $a(\lambda)$ is constant in time and so are the I_n. We have found an infinite (but countable) set of local conserved quantities. Apart an overall factor i[1], we find the following densities,

$$q_1(x) = \psi(x)\bar{\psi}(x) \tag{3.105}$$

$$q_2(x) = -i\bar{\psi}(x)\psi'(x) \tag{3.106}$$

$$q_3(x) = -\bar{\psi}(x)\psi''(x) + g\psi\psi\bar{\psi}\bar{\psi} \tag{3.107}$$

$$q_4(x) = i\left(\bar{\psi}(x)\psi'''(x) - g\bar{\psi}(x)\psi(x)\left(\psi(x)\bar{\psi}'(x) + 4\bar{\psi}(x)\psi'(x)\right)\right) \tag{3.108}$$

in the sense that $I = \int \mathrm{d}x q(x)$ is the local conserved charge. The set of conserved charges is an infinite dimensional vector space: the sum of two conserved quantities is again conserved and the same happens under scalar multiplication (that is why we could factor out the i above). Also, the Poisson brackets (3.7) make this vector space an algebra, a sub-algebra of that of observables. It appears that conserved quantities and local densities in the interacting theory are perturbations of the ones in the free theory,

$$q_n^{int} = q_n^{free} + f(g, \psi, \partial\psi, \ldots, \partial \ldots \partial\psi) \tag{3.109}$$

with,

$$\lim_{g\to 0} f(g, \psi, \partial\psi, \ldots, \partial \ldots \partial\psi) = 0 \tag{3.110}$$

3.3.3 Dynamics of Transition Coefficients

Until this moment we have not shown how the coefficients of the transfer matrix evolve in time. Indeed, one of the most striking properties of integrable models is that the time evolution of these quantities is trivial. This will also let us to exactly diagonalize the Hamiltonian. Consider (3.20b): in the rapidly decreasing case, as $x \to \pm\infty$

$$V_\infty = \frac{i\lambda^2}{2}\sigma_3 \tag{3.111}$$

so that,

$$\partial_t F = \frac{i\lambda^2}{2}\sigma_3 F \tag{3.112}$$

[1] This will be important later, when we will diagonalize the hamiltonian.

(3.33) gives the equation of motion for the transfer matrix in this limit and so for its coefficients,

$$\partial_t \mathbb{T}_\lambda(t) = i\lambda^2[\sigma_3, \mathbb{T}_\lambda] \tag{3.113}$$
$$\dot{a}(\lambda, t) = 0 \tag{3.114}$$
$$\dot{b}(\lambda, t) = i\lambda^2 b(\lambda, t) \tag{3.115}$$

with solutions,

$$a(\lambda, t) = a(\lambda, 0) \tag{3.116}$$
$$b(\lambda) = b(\lambda, 0)e^{i\lambda^2 t} \tag{3.117}$$

3.3.4 Action-Angle Variables for the NLSE

Why Inverse Scattering Method? Basically what makes a system integrable is the possibility to write it in a convenient coordinate systems, for which the dynamics becomes trivial. These special coordinates are calle *action-angle variables* and are common in the theory of Hamiltonian systems [32, 78]. The Inverse Scattering Method is the analogue of a Fourier Transform for a rather special class of non linear PDE. We now show that the Inverse Scattering Method provides a way to write the Hamiltonian in terms of action-angle variables. Suppose the function $a(\lambda)$ has N imaginary zeros $i\chi_j$. These can be shown to be always simple zeros and can be related to the bound states. The function,

$$\tilde{a}(\lambda) = a(\lambda)\prod_{j=1}^{N}\frac{\lambda + i\chi_j}{\lambda - i\chi_j} \tag{3.118}$$

is analytical[2] in the upper half plane and has no zeros. Then,

$$0 = \int_{\mathbb{R}} \frac{\log \tilde{a}(\lambda')}{\lambda' - \lambda + i\epsilon} d\lambda' \tag{3.119}$$

by Cauchy Theorem of complex analysis. By definition $|\tilde{a}(\lambda)| = |a(\lambda)|$ for $\lambda \in \mathbb{R}$ and so,

$$0 = \int_{\mathbb{R}} \frac{\log |a(\lambda')|}{\lambda' - \lambda + i\epsilon} d\lambda' + i\int_{\mathbb{R}} \frac{\arg \tilde{a}(\lambda')}{\lambda' - \lambda + i\epsilon} d\lambda' \tag{3.120}$$

Letting $\epsilon \to 0^+$ and using the well known relation,

$$\lim_{\epsilon \to 0^+} \frac{1}{x \pm i\epsilon} = \mathcal{P}\left(\frac{1}{x}\right) \mp i\delta(x) \tag{3.121}$$

we find,

$$\arg \tilde{a}(\lambda) = -\frac{1}{\pi}\mathcal{P}\int_{\mathbb{R}} \frac{\log |a(\lambda')|}{\lambda' - \lambda} d\lambda' \tag{3.122}$$

[2] See Appendix A.1

Now by deifinition,

$$\begin{aligned}\log \tilde{a}(\lambda) &= \log|\tilde{a}(\lambda)| + i \arg \tilde{a}(\lambda) \\ &= \log|a(\lambda)| + i \arg \tilde{a}(\lambda) \\ &= \log|a(\lambda)| + i \arg a(\lambda) + \sum_{j=1}^{N} \log \frac{\lambda + i\chi_j}{\lambda - i\chi_j}\end{aligned} \tag{3.123}$$

which gives,

$$\arg \tilde{a}(\lambda) = \frac{1}{i} \sum_{j=1}^{N} \log \frac{\lambda + i\chi_j}{\lambda - i\chi_j} + \arg a(\lambda) \tag{3.124}$$

Inserting in (3.122) the modulus of a plus the discrete spectrum completely determine the argument,

$$\arg a(\lambda) = -\frac{1}{i} \sum_{j=1}^{N} \log \frac{\lambda + i\chi_j}{\lambda - i\chi_j} - \frac{1}{\pi} \mathcal{P} \int_{\mathbb{R}} \frac{\log|a(\lambda')|}{\lambda' - \lambda} d\lambda' \tag{3.125}$$

In the case of NLSE in the repulsive interaction regime there are no bound states. Thus, the dispersion relation reduces to,

$$\arg a(\lambda) = -\frac{1}{\pi} \mathcal{P} \int_{\mathbb{R}} \frac{\log|a(\lambda')|}{\lambda' - \lambda} d\lambda' \tag{3.126}$$

This relation is extremely important since it allows the identification of the action variable. Indeed,

$$\begin{aligned}\log a(\lambda) &= \log|a(\lambda)| + i \arg a(\lambda) \\ &= \log|a(\lambda)| - \frac{i}{\pi} \mathcal{P} \int_{\mathbb{R}} \frac{\log|a(\lambda')|}{\lambda' - \lambda} d\lambda' \\ &= -\lim_{\epsilon \to 0^+} \frac{i}{\pi} \int_{\mathbb{R}} \frac{\log|a(\lambda')|}{\lambda' - \lambda - i\epsilon} d\lambda'\end{aligned} \tag{3.127}$$

Where in the last line we have applied again (3.121). Using the geometric series exapnsion we find a suggestive expression for the conserved charges,

$$\log a(\lambda) = \sum_{n=1}^{+\infty} \lambda^{-n} \tilde{I}_{n-1} \tag{3.128}$$

$$\tilde{I}_n = \frac{i}{\pi} \int_{\mathbb{R}} \lambda^n \log|a(\lambda)| d\lambda \tag{3.129}$$

The Hamiltonian is then,

$$H = \int_{\mathbb{R}} J(\lambda)\omega(\lambda)d\lambda \tag{3.130}$$

$$J(\lambda) = \frac{1}{g\pi}\log|a(\lambda)| \tag{3.131}$$

$$\omega(\lambda) = \lambda^2 \tag{3.132}$$

In particular (3.131) and (3.132) are obtained comparing the two expansions for $\log a(\lambda)$ in (3.101) and (3.128). Indeed, we have,

$$gI_3 = \tilde{I}_2 \tag{3.133}$$

which means,

$$\frac{i}{\pi}\int \mathrm{d}\lambda\lambda^2 \log|a(\lambda)| = ig\int \mathrm{d}x\left\{-\bar{\psi}(x)\partial_x^2\psi(x) + g|\psi(x)|^4\right\} \tag{3.134}$$

From this expression we interpret the spectral parameter λ as momentum. Thic can actually be seen deriving the Schrödinger equation from the scattering system (3.20). The last observation is that the energy of the system is always non-negative, as it should be in the repulsive case, as clear from the representation (3.130). The pair (J, ω) are the sought action-angle variables. Every conserved charge is obtained by the action varible J,

$$\mathcal{Q}_n = \int \mathrm{d}\lambda J(\lambda)q_n(\lambda) \tag{3.135}$$

where $q_n(\lambda) = \lambda^n$ is called *eigenvalue* associated to the charge. Note also that the charges as *local*. For other models the eigenvalue and the action variable are different, but relations between them remain unchanged. For example in the Sinh-Gordon model they are [9],

$$J(\theta) = \frac{8}{\pi g^2}\log|a(\lambda(\theta))| \tag{3.136}$$

$$\omega(\theta) = m\cosh(\theta) \tag{3.137}$$

with θ being the rapidity and $\lambda = me^{\theta}$. The Hamiltonian has been completely diagonalized thanks to the Inverse Scattering Method. The determination of explicit formulas for particular solutions requires to solve the inverse problem and we will not develop it here.

Comparison with the Free Theory

Again, we make contact with the free theory, to see how the integrable structure arises by deforming the simpler case. Consider the Hamiltonian (1.37) and the associated Schrödinger equation. The linearity allows to solve the problem using the Fourier Transform. Defining,

$$\tilde{\psi}(k,t) = \frac{1}{\sqrt{2\pi}}\int dx e^{-ikx}\psi(x,t) \tag{3.138}$$

we get,

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int dk A(k) e^{ikx - ik^2 t} \tag{3.139}$$

with,

$$A(k) = \frac{1}{\sqrt{2\pi}} \int dx \psi(x,0) \tag{3.140}$$

Substituting in the Hamiltonian (1.37) we get,

$$H = \frac{1}{2\pi} \int dk |A(k)|^2 k^2 \tag{3.141}$$

We see that the Hamiltonian is diagonal and we can read action-angle variables,

$$J(k) = \frac{1}{2\pi} |A(k)|^2 \tag{3.142}$$

$$\omega(k) = k^2 \tag{3.143}$$

Consider now (3.79). Clearly, $|\tilde{\psi}(\lambda)|^2 = |A(\lambda)|^2$ so that,

$$\lim_{g \to +\infty} \frac{1}{\pi g} \log |a(\lambda)| = |A(\lambda)|^2 \tag{3.144}$$

This is an interesting check since it means that the limit of the action variable in g is smooth. Formula (3.79) has also an interesting interpretation: while in the free theory the conserved charges are the modes occupations, and each single mode affect the dynamics by its own, in the interacting theory every mode talks to all the others and also in a non local way (as evident from the term with the principal value).

3.4 The NLSE on the circle: periodic case

The scattering problem anlyzed in the last section restricted the solution of the NLSE to the Schwartz class. Since the solution of the NLSE plays the role of the potential in the equivalent Schrödinger problem, the large distance boundary conditions for the differential system (3.20a) were plane waves (3.60). The energy density of the system is written in terms of the field ψ and so it goes to zero in the thermodynamic limit. A way to implement a finite energy density is to consider a periodic field on an interval $[0, L]$ and take the thermodynamic limit only at the end of the calculations in the relevant equations. Despite this is intuitive, the Inverse Scattering method becomes cumbersome in the case of periodic boundary conditions. It is possible to explicitely construct the solution in terms of the Riemann-Siegel function but once the thermodynamic limit is taken this approach becomes usless [9]. A nice reference for the period scattering problem is [79]. The main purpose of this section is to illustrate how different the periodic problem is with respect to that on the whole line and to extract the classical root density from the transfer matrix. The identification, that will become apparent and later confirmed

in several ways, for the root density, in the thermodynamic limit, will be,

$$\rho(\lambda) = \frac{J(\lambda)}{L} = \frac{1}{g\pi L} \log |a(\lambda)| \tag{3.145}$$

3.4.1 Scattering in Periodic Potential

To enlighten the basic differences which arise in the scattering problem in presence of a periodic potential we may consider the most basic problem. Consider the scattering problem introduce previously,

$$\partial_x F = U_\lambda F \tag{3.146}$$

this time the field ψ is taken to be periodic in $I = [0, L]$ with all its derivatives. In this case the above system is the one dimensional problem of a particle in a periodic potential. Given a point $x_0 \in I$ it is possible to specify a normalized basis of solutions as,

$$\zeta_+(x = x_0, x_0, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \zeta_-(x = x_0, x_0, \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.147}$$

The consequenche of the periodicity of the potential is that the solutions outside I can be obtained by a simple matrix multiplication,

$$\begin{pmatrix} \zeta_+(x + mL, x_0, \lambda) & \zeta_-(x + mL, x_0, \lambda) \end{pmatrix} = \begin{pmatrix} \zeta_+(x, x_0, \lambda) & \zeta_-(x, x_0, \lambda) \end{pmatrix} \mathcal{T}_\lambda^m \tag{3.148}$$

where m is an integer. The matrix $\mathcal{T}_\lambda$ is called *periodic transfer matrix* and is the analog of (3.66) for the periodic solution. A physical solution should be bounded so that what matters here are the eigenvalues of $\mathcal{T}_\lambda$. The involution property (3.49) remains unaltered since depends only on the form of the matrix U_λ. From this it follows that in this case too the transfer matrix is unimodular. In this case, this implies that the eigenvlues are pure phases $\rho_\pm$, otherwise the solutions blow up. By the same reasonings which led to (3.33) its trace,

$$\Delta(\lambda) \equiv \operatorname{Tr} \mathcal{T}_\lambda \tag{3.149}$$

is time independent. Since the characteristic polynomial is,

$$C(\lambda) = \lambda^2 - \lambda \operatorname{Tr} \mathcal{T}_\lambda + \det \mathcal{T}_\lambda \tag{3.150}$$

$$= \lambda^2 - \lambda \Delta(\lambda) + \rho_+ \rho_- \tag{3.151}$$

we can write its eigenvalues as,

$$\Delta(\lambda) = \frac{1}{2} \left[\Delta(\lambda) \pm \sqrt{\Delta^2(\lambda) - 4} \right] \tag{3.152}$$

The function Δ is called *discriminant* in this conext. The unimodularity condition becomes,

$$\Delta(\lambda) \in \mathbb{R}, \quad \Delta(\lambda) \leq 2 \tag{3.153}$$

At this point there are three possibilities,

1. If $\Delta(\lambda) \geq 2$ eigenvalues are real with non unitary modulus thus the solutions are not bounded. This case is excluded.

2. If $\Delta(\lambda) < 2$ eigenvalues are pure phases and the solutions outside I are obtained by a simple phase multiplication.

3. If $\Delta(\lambda) = 2$ and the roots of λ_n satisfying this equation are not degenerate (i.e. $\Delta'(\lambda) \neq 0$) the associated eigenvectors will be periodic ($\Delta(\lambda) = 2$) and antiperiodic ($\Delta(\lambda) = -2$).

The roots λ_n such that $\Delta(\lambda_n) = 0$ is called *simple spectrum*, see Fig. 3.1 and its determination allows to construct solutions in the periodic problem. Setting $x = x_0$ in (3.148) we find that,

$$\mathcal{T}_\lambda = \begin{pmatrix} \zeta_{+,1}(x_0 + L, x_0, \lambda) & \zeta_{+,2}(x_0 + L, x_0, \lambda) \\ \zeta_{-,1}(x_0 + L, x_0, \lambda) & \zeta_{-,2}(x_0 + L, x_0, \lambda) \end{pmatrix} \tag{3.154}$$

so that the discriminat can be expressed as,

$$\Delta(\lambda) = \zeta_{+,1}(x_0 + L, x_0, \lambda) + \zeta_{-,2}(x_0 + L, x_0, \lambda) \tag{3.155}$$

3.4.2 Infinite Gap Solution

For completeness we should mention how, given the simple spectrum, the Inverse Scattering Method allows to write an elegant and closed formula for the solution to the NLSE with periodic boundary conditions, and why this is a problem when we take the thermodynamic limit: this is called *infinte gap solution*. We address the reader willing to fill details to Ref. [9, 32, 33, 74, 80] and references therein. The construction is as follows. The eigenvalues of the periodic transfer matrix (3.148) are expressed in terms of,

$$y(E) = \sqrt{\Delta^2(E) - 4} \tag{3.156}$$

with $E = \sqrt{\lambda}$. This functions, due to the non trivial structure of $\Delta(E)$, see Fig. 3.1, has branch cuts in the complex E plane but it is possible to choose these cuts so that they start and end at the same points of the simple spectrum. In the simplifying hypotesis that the simple spectrum is given and its cardinality of is finite , say $2n$ the most general solution of the NLSE can be constructed. Putting into the game algebraic-geometric methods one finds that the relevant quantities are finite and are given by the $n \times n$

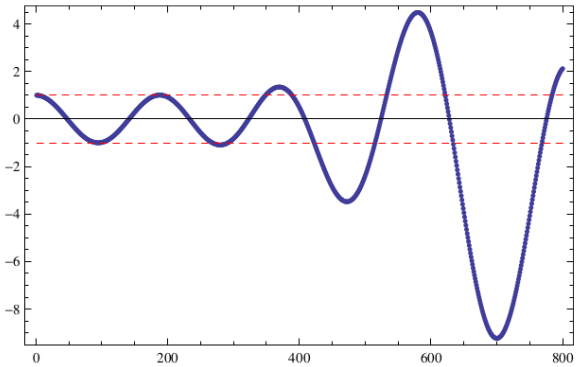

FIGURE 3.1: Typical plot of $\Delta(\lambda)$.

matrices ($i, j = 1, \ldots, n$),

$$\alpha_{ij} = 2\int_{E_{2j-1}}^{E_{2j}} \frac{E^{i-1}}{y(E)} \mathrm{d}E \quad \beta_{ij} = 2\int_{E_{2j-1}}^{E_{2j-1}} \frac{E^{i-1}}{y(E)} \mathrm{d}E \tag{3.157}$$

from which one constructs the so-called *period matrix*,

$$B = -\alpha^{-1}\tilde{\beta} \tag{3.158}$$

with $\tilde{\beta}_{ij} = \sum_{k=1}^{j} \beta_{ik}$. In term of these quantities the most general solution is [81, 82],

$$\psi(x,t) = S\frac{\Theta(\mathbf{l}|B)}{\Theta(\mathbf{k}|B)} \exp(i\Phi(x,t)) \tag{3.159}$$

where Θ is the Riemann-Siegel Theta function defined for arbitrary vector $\mathbf{v}$ and matrix M by,

$$\Theta(\mathbf{v}|M) = \sum_{\mathbf{w}\in\mathbb{Z}^n} \exp(i\pi\mathbf{w}^T M\mathbf{w} + 2\pi i\mathbf{w}^T\mathbf{v}) \tag{3.160}$$

and $\mathbf{l}$ and $\mathbf{k}$ are two $n \times n$ vectors with linear time evolution constructed out of the matrix α, $S \in \mathbb{C}$ is a complex number and the function $\Phi(x,t)$ is linear in time too. The complexity of the formula for the solution is evident: indeed, even if its implementation is hard for small values of n, in the thermodynamic limit the roots of $\Delta(\lambda) = 0$ accumulate and form a distribution. It becomes thus impossible to obtain the solution to the periodic NLS in the large volume limit and for any time, in order to study the properties of the steady state. The way out of this problem is explained in the next paragraph adn relies on the relation between the whole line and periodic problems in the thermodynamic limit.

3.4.3 Identification of the Classical Root Density

To determine the relation between the problem when the field are supposed to rapidly vanish at infiniy and that when they are defined on the circle we can think as follows. We can always associate to a periodic field $\psi(x,t)$ solution of the NLSE a compact support field,

$$\psi_L(x,t) = \begin{cases} \psi(x,t) & x \in [x_0, x_0 + L] \\ 0 & otherwise \end{cases} \tag{3.161}$$

Since the field ψ_L vanishes outside $[x_0, x_0 + L]$ it must be a linear combination of the whole line asymptotyc solutions $\psi_\lambda^\pm$ given in eq. (3.56)-(3.57). From the normalization condition (3.147) we find,

$$\zeta_\pm(x, x_0, \lambda) = \psi_\lambda^\pm \tag{3.162}$$

so that using (3.155) and (3.68),

$$\Delta(\lambda) = 2|a(\lambda)|\cos(\lambda L - \arg a(\lambda)) \tag{3.163}$$

This representation of the discrimintant will be enlightening in a moment. For the moment note that while in the rapidly decreasing case not only the trace of the transfer matrix was a constant of motion but the full coefficient $a(\lambda)$. This happend in (3.97) the fields in the definition of the matrix $Z(x,y)$ defined in (3.85) were vanishing at infinity. Here, repeating the same procedure which led to the identification of,

$$i\phi_\lambda = \log(a(\lambda)) \tag{3.164}$$

as the generating function of the conserved charges, that is solving the differential equation for the matrix $T_\lambda(x,y;t$, using also $W(x_0) = W(x_0+L)$, we arrive at,

$$\Delta(\lambda) = \operatorname{Tr} e^{Z(x_0+L)-Z(x_0)} \tag{3.165}$$

Now observe that $w_{\pm,n}$ defined in (3.87) are periodic. Indeed (3.88a)-(3.88b) shows that $w_{\pm,1}$ proportional to the field and since $w_{\pm,n}$ are determined recursively they are al polynomials in the filed, thus periodic with the same period. Thus, integrating over the period L eq. (3.94a)-(3.94b) we find that,

$$\Delta(\lambda) = 2\cos(\phi_\lambda - \lambda L) \tag{3.166}$$

This means that the generating function of conservation laws analogous to $\log(a(\lambda)$ this time is,

$$\phi_\lambda^{per} = \arccos\left(\frac{\Delta(\lambda)}{2}\right) + \lambda L \tag{3.167}$$

and expanding in powers of λ around $\lambda = 0$ and $\lambda = +\infty$ we find the conserved charges. The conserved charges found by these expansions have the same densities of that defined on the whole line in the rapidly decreasing case, the difference coming from the support of these functions. For instance,

$$N = \int_0^L \mathrm{d}x|\psi|^2 \tag{3.168}$$

$$H = \frac{1}{2}\int_0^L \mathrm{d}x|\partial_x\psi|^2 + g|\psi|^4 \tag{3.169}$$

This implies that the values of the conserved charges is not equal to that in the whole line case because thera differences coming from boundary integration terms. Equivalently, truncating the whole line field ψ to a compact support as in (3.161) introduces singularities at the boundaries. However, this is not a limitation for us since for extensice the bulk of width L the charges coincide thermodynamic limit. This argument and the comparison of (2.133) and (3.135), led the authors of Ref. [9] to identify the classical root density as in the Sinh-Gordon model. In our model the identification reads,

$$\rho(\lambda) = \frac{1}{L}J(\lambda) = \frac{1}{g\pi L}\log|(a(\lambda)|) \tag{3.170}$$

Finally, we have succeded in the promised identification of the classical root density. Given an initial condition this function is fixed since it is time independent. This function, according to what we said in the Section 2.4, encodes all the information about the initial state. What matters is that by the Inverse Scattering Method, more precisely from

the computaiton of the transfer matrix, we can access the quantity which keeps memory of the initial conditions, thus characterizing the steady state of the system. A final remark is necessary. Despite the root density is fixed once we give the initial condition, as we already sais in Section 2.2 we will consider ensembles of initial conditions. The initial state can be any out-of-equilibrium state, thermal or GGE. Indeed, in Chapter 1 we called $n(k) = \langle |A(k)|^2 \rangle$ the the classical filling fraction. In this respect, given an inital ensemble of configurations, the root density will be,

$$\rho(\lambda) = \frac{1}{L}\langle J(\lambda)\rangle = \frac{1}{g\pi L}\langle \log|(a(\lambda)|)\rangle. \tag{3.171}$$

Chapter 4

Density Moments and Full Counting Statistics

In this Chapter we present our fundamental results. We exactly compute the moments of the local density on an arbitrary GGE. Furthermore, we provide exact expressions for the whole probability distribution of the local density. This quantity is called *full counting statistics* (GCS). In particular, the FCS is a rather imporant object from an experimental point of view. Indeed, it gives access to the full distribution of the number of particles, not only its moments (i.e. the density one point functions). In obtaining our results we exploit several tools discussed in previous chapters, namely the semiclassical limit of the LL model and the root density characterizing the steady state. First, we recall an exact mapping between the Sinh-Gordon model and the Lieb-Liniger model, existing already at the quantum level, by means of a proper non relativistic limit. After, we present the basis of our later computations, a formula due to Negro and Smirnov [43, 44]. Handling the non relativistic limit [8, 42] and the semiclassical limit [73] of the latter we can find analytical expressions for the FCS and a set of recursive equations determining the one point functions. The latters are obtained by means of a semiclassical limit of existing formulae for the LL model. Theoretical predictions are supported with numerical simulations whose discussion is referred to the last Chapter.

4.1 From Sinh-Gordon to Lieb-Liniger: the Non Relativistic Limit

In this section we review the fundamental derivation of the LL model as the non-relativistic limit (NR limit) of the Sinh-Gordon model as first appeared in [46]. Concerning the classical discussions on NR limits in field theory the reader may also consult [83, 84]. We present the procedure at the lagrangian level but from the correspondence of the scattering matrices it will be clear that the mapping is easily extended to the TBA equations of the model. Restoring the explicit presence of the velocity of light c ,the Sinh-Gordon lagrangian is,

$$\mathcal{L}_{SHG} = \frac{1}{2}\left(c^{-2}(\partial_t\phi)^2 - (\partial_x\phi)^2\right) - \frac{m_0^2c^2}{\gamma^2}\left(\cosh(\gamma\phi) - 1\right) \tag{4.1}$$

where ϕ is a scalar and hermitian bosonic field, γ is the coupling constant, m_0 is a mass scale related to the renormalized mass as [29],

$$m_0^2 = M^2\frac{\pi\alpha}{\sin(\pi\alpha)}, \quad \alpha = \frac{c\gamma^2}{8\pi + c\gamma^2} \tag{4.2}$$

The Lieb-Liniger model has been introduced in the first Chapter. Restoring the mass we have,

$$\mathcal{L}_{LL} = \frac{i}{2}\left(\psi^\dagger \partial_t \psi - \partial_t \psi^\dagger \psi\right) - \frac{1}{2m}|\psi|^2 - g\psi^\dagger\psi^\dagger\psi\psi \tag{4.3}$$

The first insight is furnished by the scattering matrices of the models. We have,

$$S_{SHG}(\theta,\alpha) = \frac{\sinh(\theta) - i\sin(\pi\alpha)}{\sinh(\theta) + i\sin(\pi\alpha)} \quad S_{LL}(k,g) = \frac{k - 2mig}{k + 2img} \tag{4.4}$$

where it is important to recall the θ is the rapidity parametrizing energy $E(\theta) = Mc^2\cosh(\theta)$ and momentum $P(\theta) = Mc\sinh(\theta)$. If we take the limit,

$$c \to +\infty, \quad g \to 0, \quad gc = const. \tag{4.5}$$

we obtain,

$$S_{SHG}(\theta,\alpha) \to \frac{\frac{k}{Mc} - \frac{i\gamma^2 c}{8}}{\frac{k}{Mc} + \frac{i\gamma^2 c}{8}} \tag{4.6}$$

resulting in the identification,

$$g = \frac{c^2\gamma^2}{16}, \quad m = M \tag{4.7}$$

To perform the identification of the field content of the two theories at lagrangian level the Sinh-Gordon field is written as,

$$\phi(x,t) = \frac{1}{\sqrt{2m}}\left(\psi(x,t)e^{-im_0c^2t} + h.c.\right) \tag{4.8}$$

where *h.c.* stands for hermitian conjugate. Substituting in the lagrangian, the terms containing factors $e^{inm_0c^2t}$ with $n > 1$ higly oscillates and average to zero when integrated over a small but finite time interval and will be neglected. This means that the momentum operator is,

$$\pi(x,t) = \frac{1}{c^2\sqrt{2m_0}}\left[\left(\dot{\psi}(x,t) - im_0c^2\psi(x,t)\right)e^{-im_0c^2t} + h.c.\right] \tag{4.9}$$

$$= -i\sqrt{\frac{m_0}{2}}\left(\psi(x,t)e^{-im_0c^2t} - h.c.\right) + O\left(\frac{1}{c^2}\right) \tag{4.10}$$

Inverting we find,

$$\psi(x,t) = e^{im_0c^2t}\left(\sqrt{\frac{m_0}{2}}\phi(x,t) + \frac{i}{\sqrt{2m_0}}\pi(x,t)\right) \tag{4.11}$$

The expression for $\psi^\dagger$ is obtained by hermitian conjugation. This proves that if,

$$[\phi(x,t), \pi(y,t)] = i\delta(x-y) \tag{4.12}$$

then,

$$[\psi(x,t),\psi^\dagger(y,t)] = \delta(x-y) \tag{4.13}$$

Inserting (4.8) in the kinetic part of the Sinh-Gordon lagrangian gives,

$$\frac{1}{2m_0c^2}\partial_t\psi^\dagger\partial\psi - \frac{1}{2m_0}\partial_x\psi^\dagger\partial\psi + \frac{i}{2}\left(\psi^\dagger\partial\psi - \partial\psi^\dagger\psi\right) + \frac{1}{2}m_0c^2\psi^\dagger\psi \tag{4.14}$$

Expanding around $c = +\infty$,

$$m_0^2 = M^2 + \frac{2}{3}\frac{M^2\gamma^2}{c^2} + O(c^{-2}) \tag{4.15}$$

we see that the second term can be neglected in the NR limit. To deal with the interaction term we expand in power series and use the binomial formula to express ϕ in terms of ψ. The caveat is that each oscillating phase is to be neglected and in the binomial expansion of ϕ^{2k} only symmetric middle terms survive that is the ones proportional to $\binom{2n}{n}$. We end up with,

$$\frac{m_0c^2}{\gamma^2}(\cosh(\gamma\phi)-1) = \sum_{n=1}^{+\infty}\frac{1}{(n!)^2}\frac{m_0c^2}{\gamma^2}\left(\frac{\gamma^2}{2m_0}\right)^n(\psi^\dagger\psi)^n \tag{4.16}$$

The term $n=1$ cancels the last term in the kinetic term expansion while the term $n=2$ provides the LL quartic interaction. Other terms are seen to be negligible in the NR limit because,

$$\sum_{n=3}^{+\infty}\left[\frac{c^2(\gamma^2)^{n-1}}{2^n(n!)^2m_0^{n-2}}(\psi^\dagger\psi)^n + \dots\right] = \sum_{n=3}^{+\infty}\frac{2^{3n-4}g^{n-1}}{(n!)^2(mc^2)^{n-2}}(\psi^\dagger\psi)^n + \dots \tag{4.17}$$

ang g, the coupling in the LL model is to be kepts fixed as state in (4.5). This procedure estabilishes the correspondence between the two models. The most important result for our concerns is the correspondence between Sinh-Gordon fields and LL fields. This reads,

$$\lim_{NR}\langle :\phi^{2K+1}:\rangle = 0, \quad \lim_{NR}\langle :\phi^{2K}:\rangle = \binom{2K}{K}\frac{1}{(2m)^K}\langle(\psi^\dagger\psi)^K\rangle \tag{4.18}$$

4.2 The Negro-Smirnov Formula

In this section we present the Negro-Smirnov formula [43, 44]. The semiclassical limit of the Negro-Smirnov formula has been perfomed in Ref. [73] and we only sketch the derivation. The expressions below will be the beginning of our calculations of the full counting statistics of the density. The Negro-Smirnov formula concerns the expectation values on arbitrary states of a partucular class of local operators in the quantum Sinh-Gordon model, called *vertex operators* and can be written as,

$$\frac{\langle e^{(k+1)\gamma\phi}\rangle}{\langle e^{k\gamma\phi}\rangle} = 1 + \frac{2\sin(\pi\alpha(2k+1))}{\pi}\int \mathrm{d}\theta\frac{e^\theta}{1+e^{\epsilon_q(\theta)}}p_q^k(\theta) \tag{4.19}$$

with,

$$p_q^k(\theta) = e^{-\theta} + \int \mathrm{d}\gamma \frac{1}{1+e^{\epsilon_q(\theta)}} \chi_k(\theta) p_q^k(\theta) \tag{4.20}$$

$$\chi_k(\theta) = \frac{i}{2\pi}\left(\frac{e^{-2ik\pi\alpha}}{\sinh(\theta+i\pi\alpha)} - \frac{e^{2ik\pi\alpha}}{\sinh(\theta-i\pi\alpha)}\right) \tag{4.21}$$

We have indicated quantum quantities with a subscript and we will do this for the rest of the discussion. Classical quantities are indicated without any subscritp. The first thing to do is to estabilish the right scalings of the quantities involved in the expextation values, namely the coupling constant, the field and the inverse temperature. We have [73],

$$\beta \to \hbar\beta, \quad \gamma^2 \to \hbar\gamma^2, \quad O(\phi) \to O(\sqrt{\hbar}\phi) \tag{4.22}$$

where O is a local observable. This is easily recognized from the quantum thermal partition function in the same way we did for the LL model. To perform the semiclassical limit of the formulae above, we observe that $\alpha = O(\hbar)$. The left hand side remains unchanged since the scalings of the coupling and the field cancel each other. The first step is to deal with χ_k. Using,

$$\frac{1}{\sinh(x-i\alpha\pi)} = i\pi\delta(x) + \mathcal{P}\frac{1}{\sinh(x)} + \pi\alpha\left(\pi\delta(x)\partial_x - i\mathcal{P}\frac{1}{\sinh(x)}\partial_x\right) + O(\alpha^2) \tag{4.23}$$

in the distributional sense and inserting in the equation for p_k and defining $p_q^k(\theta) = \hbar^{-1}n(\theta)p^k(\theta)$ we get a finite expression, not dependent on $\hbar$. Note that $n(\theta)$ is the classical filling fraction. The final result is,

$$\frac{\langle e^{(k+1)\gamma\phi}\rangle}{\langle e^{k\gamma\phi}\rangle} = 1 + (2k+1)\frac{\gamma^2}{4\pi}\int \mathrm{d}\theta e^{\theta} n(\theta) p^k(\theta) \tag{4.24}$$

where,

$$p^k(\theta) = e^{-\theta} + \frac{\gamma^2}{4}\mathcal{P}\int \frac{\mathrm{d}\gamma}{2\pi}\frac{1}{\sinh(\theta-\gamma)}(2k - \partial_\gamma)(n(\gamma)p^k(\gamma)) \tag{4.25}$$

4.3 Exact One Point Functions

The one point functions of the LL model have been computed in in Ref. [8, 42], which generalize Ref. [47], but the derivation is rather involved so that we refer to the aforementioned literature for the technical details involved[35, 85]. One points functions for the NLSE can be obtained in two different ways: on one hand, we can do the semicalssical limit of the known quantum expressions; on the other hand, we may start from the classical Negro-Smirnov formula and do the non relativistic limit; both methods are consistent: we take the first route for the one point functions and the second for the full counting statistics of the density. One point functions for the quantum LL model reads,

$$g_K(x) = \frac{\mathcal{O}_K(x)}{n^K} = \frac{\langle\rho|\,(\psi^\dagger(x))^K(\psi(x))^K\,|\rho\rangle}{n^K} \tag{4.26}$$

where the density n is defined by,

$$n \equiv \lim_{TH} N/L = \int_{-\infty}^{+\infty} d\lambda \rho(\lambda) \tag{4.27}$$

In the quantum case they have the form,

$$\mathcal{O}_K = (K!)^2 (mg)^K \sum_{n_j : \sum_j j n_j = K} \prod_j \left[\frac{1}{n_j!} \left(\frac{\mathcal{B}_j}{2\pi mg} \right)^{n_j} \right] \tag{4.28}$$

where,

$$\mathcal{B}_j = \frac{1}{j} \int_{-\infty}^{+\infty} d\lambda n(\lambda) b_{2j-1}(\lambda) \tag{4.29}$$

The functions b_j are determined recursively,

$$b_{2n}(\lambda) = \int_{-\infty}^{+\infty} \frac{d\mu}{2\pi} \theta(\mu) \left\{ \varphi(\lambda - \mu) \left[b_{2n}(\mu) - b_{2n-2}(\mu) \right] + \Gamma(\lambda - \mu) \left[2 b_{2n-1}(\mu) - b_{2n-3}(\mu) \right] \right\} \tag{4.30a}$$

$$b_{2n+1}(\lambda) = \delta_{n,0} + \int_{-\infty}^{+\infty} \frac{d\mu}{2\pi} \theta(\mu) \left\{ \varphi(\lambda - \mu) \left[b_{2n+1}(\mu) - b_{2n-1}(\mu) \right] + \Gamma(\lambda - \mu) b_{2n}(\mu) \right\} \tag{4.30b}$$

$$b_{j \leq 0}(\lambda) = 0 \tag{4.30c}$$

Also, it appears the filling fraction,

$$n(\lambda) = \frac{1}{e^{\epsilon(\lambda)} + 1} \tag{4.31}$$

and the LL kernel φ. The function Γ is given by,

$$\Gamma(\lambda) = \frac{2\lambda}{\lambda^2 + (2mg)^2} \tag{4.32}$$

In order to get the same quantities for the NLSE we have to perform the semiclassical limit of these formulae. The scalings we will use are, again, given in (1.185). All the formulae above are "quantum" and we have omitted the subscripts for notational simplicty. In the following, we do not label classical quantities while we label quantum ones. We immediately find,

$$\Gamma_q(\lambda) \mapsto \Gamma(\lambda) = \frac{2\lambda}{\lambda^2 + (2mg\hbar)^2} \tag{4.33}$$

$$n_q(\lambda) \mapsto \theta(\lambda) = \frac{1}{e^{\log(\hbar \epsilon(\lambda))} + 1} = 1 - \hbar \epsilon(\lambda) + O(\hbar^2) \tag{4.34}$$

$$\mathcal{O}_K = \hbar^K \mathcal{O}_{K,q} \tag{4.35}$$

while $\varphi(\lambda)$ is given in (2.139). Eqs. (4.30a)-(4.30b) respectively become,

$$b_{2n,q}(\lambda) = \int_{-\infty}^{+\infty} \frac{d\mu}{2\pi}(1 - \hbar\epsilon(\mu) + O(\hbar^2)) \times \{\frac{4mg\hbar}{(\lambda-\mu)^2 + (2mg\hbar)^2}\left[b_{2n,q}(\mu) - b_{2n-2,q}(\mu)\right] + \frac{2(\lambda-\mu)}{(\lambda-\mu)^2 + (2mg\hbar)^2}\left[2b_{2n-1,q}(\mu) - b_{2n-3,q}(\mu)\right]\} \tag{4.36a}$$

$$b_{2n+1,q}(\lambda) = \delta_{n,0} + \int_{-\infty}^{+\infty} \frac{d\mu}{2\pi}(1 - \hbar\epsilon(\mu) + O(\hbar^2)) \times \{\frac{4mg\hbar}{(\lambda-\mu)^2 + (2mg\hbar)^2}\left[b_{2n+1,q}(\mu) - b_{2n-1,q}(\mu)\right] + \frac{2(\lambda-\mu)}{(\lambda-\mu)^2 + (2mg\hbar)^2} b_{2n,q}(\mu)\} \tag{4.36b}$$

We will employ the following expressions,

$$\frac{\hbar}{x^2+\hbar^2} = \pi\delta(x) - \hbar\mathcal{P}\left(\frac{1}{x}\partial_x\right) + O(\hbar^2) \tag{4.37}$$

$$\frac{x\hbar}{x^2+\hbar^2} = \pi x\delta(x) + \hbar\mathcal{P}\left(\frac{1}{x}\right) + O(\hbar^2) = \hbar\mathcal{P}\left(\frac{1}{x}\right) + O(\hbar^2) \tag{4.38}$$

$$\frac{\hbar^2}{x^2+\hbar^2} = \hbar\pi\delta(x) - \hbar^2\mathcal{P}\left(\frac{1}{x}\partial_x\right) + O(\hbar^3) \tag{4.39}$$

$$\lim_{\hbar\to 0}\frac{x}{x^2+\hbar^2} = \mathcal{P}\left(\frac{1}{x}\right) + O(\hbar^2) \tag{4.40}$$

in the distributional sense[1]. Before doing the general case we may analize the simplest one. For $K = 1$, formula (4.28) and the scalings (1.185) gives,

$$O_1 = \frac{\hbar\mathcal{B}_{1,q}}{2\pi} \tag{4.41}$$

This means that we must have $\mathcal{B}_{1,q} = O(\hbar^{-1})$. But from (4.36b)-(4.30c) with $n \mapsto n-1$, putting $n = 1$, we get,

$$b_{1,q}(\lambda) = 1 + \int_{-\infty}^{+\infty}\frac{d\mu}{2\pi}(1 - \hbar\epsilon(\mu) + O(\hbar^2))\frac{4mg\hbar}{(\lambda-\mu)^2+(2mg\hbar)^2}b_{1,q}(\mu)$$
$$\overset{\hbar\to 0}{=} 1 + b_{1,q}(\lambda) - 4\hbar mg\mathcal{P}\int_{-\infty}^{\infty}\frac{d\mu}{2\pi}\frac{\partial_\mu b_{1,q}(\mu)}{\lambda-\mu} - \hbar\epsilon(\lambda)b_{1,q}(\lambda) + O(\hbar^2) \tag{4.42}$$

which gives,

$$\hbar\epsilon(\lambda)b_{1,q}(\lambda) = 1 - 4\hbar mg\mathcal{P}\int_{-\infty}^{\infty}\frac{d\mu}{2\pi}\frac{\partial_\mu b_{1,q}(\mu)}{\lambda-\mu} + O(\hbar^2) \tag{4.43}$$

[1]The proof is easy going in Fourier space.

Then, $\mathcal{B}_{1,q} = O(\hbar^{-1})$ because $\theta(\lambda) = 1 + O(\hbar)$. To understand the scaling of the $\mathcal{B}_{j,q}$ and $b_{j,q}$ with $\hbar$, let us see what happens for $\mathcal{O}_2$. (4.28) gives,

$$\mathcal{O}_{2,q} = g_q^2 \left(\frac{\mathcal{B}_{1,q}^2}{2\pi^2 g_q^2} + \frac{\mathcal{B}_{2,q}}{\pi g_q} \right) \tag{4.44}$$

Rescaling,

$$\mathcal{O}_2 = \hbar^2 \frac{\mathcal{B}_{1,q}^2}{2\pi^2} + \hbar^3 \frac{\mathcal{B}_{2,q}}{\pi} \tag{4.45}$$

To get a finite expression we must have $\mathcal{B}_{2,q} = \hbar^{-3}\mathcal{B}_2$. At this point the scaling is clear:

$$\mathcal{B}_{j,q} = \hbar^{1-2j}\mathcal{B}_j \tag{4.46}$$

Direct substitution in (4.28) shows that this is the case and we find an $\hbar$-independent formula. An immediate consequence is that,

$$b_{j,q} = \hbar^{-j} b_j \tag{4.47}$$

Using the above scaling in (4.30a)-(4.30b) leads to equations similar to (4.36a)-(4.36b). We keep first order in $\hbar$:

$$\begin{aligned} b_{2n,q}(\lambda) = \int_{-\infty}^{+\infty} \frac{d\mu}{2\pi} (1 - \hbar\epsilon(\mu) + O(\hbar^2)) \\ \times \{ \frac{4mg\hbar}{(\lambda-\mu)^2 + (2mg\hbar)^2} \left[b_{2n,q}(\mu) - \hbar^2 b_{2n-2,q}(\mu) \right] \\ + \frac{2(\lambda-\mu)}{(\lambda-\mu)^2 + (2mg\hbar)^2} \left[\hbar 2 b_{2n-1,q}(\mu) - \hbar^3 b_{2n-3,q}(\mu) \right] \} \end{aligned} \tag{4.48}$$

The terms proportional to $b_{2n-2,q}$ and $b_{2n-3,q}$ gives higer order contributions and we neglect them. Thus,

$$\begin{aligned} b_{2n}(\lambda) = \int_{-\infty}^{+\infty} \frac{d\mu}{2\pi} (1 - \hbar\epsilon(\mu) + O(\hbar^2)) \\ \times \{ \frac{4mg\hbar}{(\lambda-\mu)^2 + (2mg\hbar)^2} b_{2n}(\mu) \\ + \frac{2\hbar(\lambda-\mu)}{(\lambda-\mu)^2 + (2mg\hbar)^2} 2 b_{2n-1}(\mu) \} \end{aligned} \tag{4.49}$$

At order $O(\hbar)$ we find,

$$b_{2n}(\lambda) = b_{2n}(\lambda) - 4\hbar mg\mathcal{P} \int_{-\infty}^{+\infty} \frac{d\mu}{2\pi} \frac{\partial_\mu b_{2n}(\mu)}{\lambda - \mu} + 2\hbar\mathcal{P} \int_{-\infty}^{+\infty} \frac{d\mu}{2\pi} \frac{2 b_{2n-1}(\mu)}{\lambda - \mu} - \hbar\epsilon(\lambda) b_{2n}(\lambda) \tag{4.50}$$

Simplifing everything we get a finite expression which determines b_{2n},

$$\epsilon(\lambda)b_{2n}(\lambda) = 2\mathcal{P}\int_{-\infty}^{+\infty}\frac{d\mu}{2\pi}\frac{2b_{2n-1}(\mu)}{\lambda-\mu} - 4mg\mathcal{P}\int_{-\infty}^{+\infty}\frac{d\mu}{2\pi}\frac{\partial_\mu b_{2n}(\mu)}{\lambda-\mu} \tag{4.51}$$

For b_{2n+1}, $n > 0$, the procedure, and also the structure of the equation, is the same. This result implies that in the classical case we can write equations for b's in compact from as,

$$\epsilon(\lambda)b_n(\lambda) = \delta_{n,1} + 2p_n\mathcal{P}\int_{-\infty}^{+\infty}\frac{d\mu}{2\pi}\frac{b_{n-1}(\mu)}{\lambda-\mu} - 4mg\mathcal{P}\int_{-\infty}^{+\infty}\frac{d\mu}{2\pi}\frac{\partial_\mu b_n(\mu)}{\lambda-\mu} \tag{4.52}$$

with $p_n = 1$ if n is odd and $p_n = 2$ if n is even. To verify the correctness of (4.26) in the semiclassical limit we simulated the theory on a computer and solved numerically the NLSE for many initial states generated according to a free theory probablity distribution. Thus, we quenched from $g_i = 0$ to $g_f = 1$, for different values of the density. In such a way we explored a parameter region where the interaction term is predominant. Indeed, the ratio between the kinetic and potential terms ot the Hamiltonian is proportional to the density: the higer the density the stronger the interaction. To be quantitative the ratio is,

$$r = \frac{2m\lambda}{D\hbar^2} \tag{4.53}$$

where D is the density. This ratio measure the effective strength of the interaction at zero temperature. Indeed, if $T > 0$ there is another length scale into the game, namely the De Broglie wavelength. Adding a chemical potential to the Hamiltonian introduces a further length scale. As we have more constraints, like in a GGE, we have other length scales in the problem. Thus, we take as a measure of the interaction uniquely the density, since on the line the more are the particles per unit length the more they cannot avoid themselves. In Figure 5.1 we report the values of $O_K = \langle(\bar{\psi}\psi)^K\rangle$ for different values of K as functions of the lattice spacing used to regularize the theory. As can be seen, in our findings, there is a quadratic convergence towards the continuum value. Techincal aspects on the simulation can be found in the relative chapter. The solution of the linear integral equations (4.52) is not difficult, they are recursive in nature and require the computation of a finite number of integrals. In Figure 5.5 the same extrapolation procedure has been done to estimate the continuum value for the values of O_k: this is necessary because the input fed into (4.52) is the *pseudoenergy*, which in turn is computed from the numerical estimate of the root density via $\epsilon(\lambda) = 1/\rho(\lambda)$ as in (2.151). To complete the analysis, in Figure 5.7 we see the the relative percentage error between theoretical data obtained solving the equations and direct simulation. We find excellent agreement for low K powers wtih an error growing as in Figure 5.8.

4.4 Density Full Counting Statistics

In this Section we compute the *Full Counting Statistics* (FCS) for the NLS model. This is simply the probability distribution of the density,

$$P(\lambda) = \langle\delta(\lambda - \bar{\psi}\psi)\rangle \tag{4.54}$$

In the quantum realm the computation of a similar quantity, namely the full counting statistics for the number of particle in a finite but small interval is reported in Ref. [42], exploiting again the Negro-Smirnov conjecture. The result obtained there can be use to compute the probability to find a certain number of particles in an interval of length $\Delta \ll D^{-1}$, where D is the density of particles. In Ref. [86] the same quantity is computed in specific approximations.

What we do here is to compute the full counting statistics of the *density* of particles. Inserting $\delta(x) = \frac{1}{2\pi}\int \mathrm{d}k e^{ikx}$ in (4.54), we define the generating function as,

$$G(\gamma) = \langle e^{i\gamma\bar{\psi}\psi}\rangle \tag{4.55}$$

from which we get $P(\lambda)$,

$$P(\lambda) = \int \frac{\mathrm{d}\gamma}{2\pi} G(\gamma) e^{-i\gamma\lambda} \tag{4.56}$$

The generating function G can be computed exactly using the Negro-Smirnov conjecture and the NR limit correspondence between ShG and LL models. Let us start this route.

4.4.1 Generating function through NR limit

The starting point is the classical version of Smirnov-Negro (SN). The classical ShG model with c restored is ruled by the following action,

$$\mathcal{L} = \int \mathrm{d}x \, \frac{1}{2c^2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_x\phi)^2 - \frac{m^2c^2}{\gamma^2}\cosh(\gamma\phi) \tag{4.57}$$

The NLS has hamiltonian,

$$H = \int \mathrm{d}x \, \frac{1}{2m}\partial_x\psi^\dagger\partial_x\psi + g|\psi|^4 \tag{4.58}$$

The NR limit is attained sending $c \to \infty$ posing

$$\gamma^2 = 16g/c^2 \tag{4.59}$$

The correspondence at the level of observables is (no need of normal ordering in the classical case)

$$\lim_{\mathrm{NR}}\langle\phi^{2n+1}\rangle = 0 \qquad \lim_{\mathrm{NR}}\langle\phi^{2n}\rangle = \binom{2n}{n}\frac{1}{(2m)^n}\langle|\psi|^{2n}\rangle \tag{4.60}$$

Expectations value are on the GGEs of the ShG and LL respectively. At the level of TBA the limit is just a small rapidity limit: on the filling fractions n (that is the only ingredient we need to know), the correspondence is

$$n_{\mathrm{LL}}(p) \simeq n_{\mathrm{ShG}}(p/(mc)) \tag{4.61}$$

We just recall that the filling is related to the root density $\rho(\lambda)$ by mean of $n(\lambda) = 2\pi\rho(\lambda)/(\partial_\lambda p)^{\mathrm{dr}}$, with p the momentum which has to be properly dressed. The dressing

and TBA for arbitrary GGE's in Lieb Liniger have been derived in Chapter 2.

The classical Negro-Smirnov formula (with c restored) states

$$\frac{\langle e^{(k+1)\gamma\phi)}\rangle}{\langle e^{k\gamma\phi)}\rangle} = 1 + (2k+1)\frac{c\gamma^2}{4\pi}\int \mathrm{d}\theta e^{\theta} n_{\mathrm{ShG}}(\theta) p_k(\theta) \tag{4.62}$$

with $p_k(\theta)$ satisfying,

$$p_k(\theta) = e^{-\theta} + \frac{c\gamma^2}{4}\mathcal{P}\int \frac{\mathrm{d}\mu}{2\pi}\frac{1}{\sinh(\theta-\mu)}(2k-\partial_\mu)(n_{\mathrm{ShG}}(\mu)p_k(\mu)) \tag{4.63}$$

Now we need to take the proper limit of the Negro-Smirnov formula. First of all, we deal with the l.h.s. and use Eq. (4.59), we get

$$\frac{\langle e^{(k+1)4\sqrt{g}c^{-1}\phi)}\rangle}{\langle e^{4k\sqrt{g}c^{-1}\phi)}\rangle} \tag{4.64}$$

A limit $c\to\infty$ with k fixed is trivial, following Ref. [42], we set $q = kc^{-1}$ and consider $c\to\infty$ keeping q fixed. Then we get

$$\frac{\langle e^{(qc+1)4\sqrt{g}c^{-1}\phi)}\rangle}{\langle e^{q4\sqrt{g}\phi)}\rangle} \simeq 1 + \frac{1}{c}\lim_{\mathrm{NR}}\partial_q \log\langle e^{q4\sqrt{g}\phi)}\rangle + \dots \tag{4.65}$$

Further terms in the expansions are not needed. We see that the first term of the r.h.s. of eq. (4.64) exactly cancels the same term in formula (4.63). Thus mathcing the next term we get

$$\lim_{\mathrm{NR}}\partial_q \log\langle e^{q4\sqrt{g}\phi)}\rangle = \lim_{\mathrm{NR}}\left[(2qc+1)\frac{4g}{\pi}\int \mathrm{d}\theta e^{\theta} n_{\mathrm{ShG}}(\theta) p_{cq}(\theta)\right] \tag{4.66}$$

On the r.h.s. we now change variables letting $\theta = p/mc$, then take the limit. The jacobian gives an extra c factor that erases the one in front of the r.h.s. and the filling goes to the Lieb Liniger one. We drop the subscript, since from now on we have to deal only with the filling in the NLS.

$$\lim_{\mathrm{NR}}\left[(2qc+1)\frac{4g}{\pi}\int \mathrm{d}\theta e^{\theta} n_{\mathrm{ShG}}(\theta) p_{cq}(\theta)\right] = \frac{8g}{m\pi}\int \mathrm{d}p\, n(p) q\zeta_q(p) \tag{4.67}$$

Above, we defined

$$\zeta_q(p) = \lim_{\mathrm{NR}}(p_{cq}(p/(mc)) \tag{4.68}$$

We now take the limit of the integral equation satisfied by p_k to get the integral equation for ζ_q

$$\zeta_q(p) = 1 + 4g\mathcal{P}\int \frac{\mathrm{d}\lambda}{2\pi}\frac{1}{p-\lambda}(2q - m\partial_\lambda)(n(\lambda)\zeta_q(\lambda)) \tag{4.69}$$

Now, we take (4.66) together with (4.67), integrate both sides in q and take the exponential. Then, through (4.60) we get the generating function of the one pt functions in

NLS,

$$\lim_{\text{NR}} \langle e^{q4\sqrt{g}\phi)} \rangle = \sum_{n=0}^{\infty} q^{2n} \frac{(8g/m)^n}{(n!)^2} \langle |\psi|^{2n} \rangle = \exp\left[\frac{8g}{m\pi} \int_0^q \mathrm{d}\tilde{q} \int \mathrm{d}p\, n(p) \tilde{q} \zeta_{\tilde{q}}(p) \right] \tag{4.70}$$

where n is the filling fraction and $\zeta_{\tilde{q}}(p)$ satisfies the integral equation (4.69). Expanding both sides in q, one can recover the one point functions we got from the semiclassical limit. Here we proceed straight to compute the FCS.

4.4.2 From Negro-Smirnov to FCS

The NR limit of the classical ShG model gave us the expression (4.70). Here, we focus on the auxiliary function,

$$F(p) = \sum_{n=0}^{\infty} p^n \frac{(8g/m)^n}{(n!)^2} \langle |\psi|^{2n} \rangle \tag{4.71}$$

and we note that,

$$\partial_p^n F|_{p=0} = \frac{(8g/m)^n}{(n!)} \langle |\psi|^{2n} \rangle \tag{4.72}$$

Thus,

$$\sum_{n=0}^{\infty} \left(\frac{i\gamma}{8g/m} \right)^n \partial_p^n F(p) = \langle \sum_{n=0}^{\infty} \frac{1}{n!} (i\gamma |\psi|^2) \rangle = \langle e^{i\gamma |\psi|^2} \rangle = G(\gamma) \tag{4.73}$$

Introducing the Fourir Transform of F,

$$\tilde{F}(k) = \int \frac{\mathrm{d}p}{2\pi} F(p) e^{-ikp} \tag{4.74}$$

resumming the geometric series we find,

$$\sum_{n=0}^{\infty} \left(\frac{i\gamma}{8g/m} \right)^n \partial_p^n F|_{p=0} = \int \frac{\mathrm{d}k}{2\pi} e^{ikp} \frac{1}{1 + \frac{k\gamma}{8g/m}} \tilde{F}(k) \tag{4.75}$$

Inserting $p = 0$ and expressing $\tilde{F}$ back in real space we get,

$$G(\gamma) = \int \frac{\mathrm{d}k \mathrm{d}p'}{2\pi} \frac{1}{1 + \frac{k\gamma}{8g/m}} e^{-ikp'} F(p) \tag{4.76}$$

An immediate analytical check of our computation is the following. From the definition of $G(\gamma)$ (4.55) we see that,

$$\mu_n = \int_{-\infty}^{+\infty} P(\lambda) \lambda^n \mathrm{d}\lambda = (-i)^n \frac{\mathrm{d}^n G(\gamma)}{\mathrm{d}\gamma^n} \Big|_{\gamma=0} \tag{4.77}$$

From (4.76) we find,

$$\begin{aligned}
\frac{\mathrm{d}^n G(\gamma)}{\mathrm{d}\gamma^n}\Big|_{\gamma=0} &= \left(-\frac{m}{8g}\right)^n \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \mathrm{d}p' k^n n! e^{-ikp'} F(p') \\
&= \left(-\frac{m}{8g}\right)^n \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \mathrm{d}p' n! i^n \frac{\mathrm{d}^n}{\mathrm{d}p'^n} \left[e^{-ikp'}\right] F(p') \\
&= -\left(-\frac{m}{8g}\right)^n \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \mathrm{d}p' n! i^n \frac{\mathrm{d}^n}{\mathrm{d}p'^n} \left[F(p')\right] e^{-ikp'} \\
&= -\left(-\frac{m}{8g}\right)^n \int_{-\infty}^{\infty} \mathrm{d}p' n! i^n \delta(p') \frac{\mathrm{d}^n}{\mathrm{d}p'^n} F(p') \\
&= i^n \langle |\psi|^{2n} \rangle
\end{aligned} \tag{4.78}$$

where in the last line we have used the definition of F in (4.71). From (4.76) we can obtain the whole FCS we are interested in. There is a pole in the integrand but looking at the generating function (4.55) suggests to regularize the singularity with the prescription,

$$\gamma \to \gamma + i\epsilon \quad \epsilon > 0$$

This choice will make G bounded allowing to exchange the order of integrals and define,

$$I(\gamma, p') = \int \frac{\mathrm{d}k}{2\pi} \frac{-k^*(\gamma)}{k - k^*(\gamma)} e^{-ikp'} \tag{4.79}$$

where,

$$k^*(\gamma) = -\frac{8g}{m}(\gamma + i\epsilon)^{-1} \tag{4.80}$$

The integral is computed with residue theorem and Jordan's lemma (which ensures that the complex and the real integral coincide). Since $\epsilon \to 0^+$ we can write,

$$k^*(\gamma) = -\frac{8g}{m}\frac{\gamma - i\epsilon}{\gamma^2 + \epsilon^2} = -\frac{8g}{m}(|\gamma|^{-1} - i\xi) \quad 0 < \xi = O(\epsilon) \tag{4.81}$$

For infinitesimal and positive ϵ, $Im(k^*) > 0$ and so the pole of the integrand is always in the upper half complex plane. If $p' > 0$ we can apply Jordan's lemma in the lower half plane while for $p' < 0$ in the upper half. Then, when $p' > 0$ there are no poles and the contribution of residues is 0. On the other side we get,

$$\begin{aligned}
I(\gamma, p') &= \theta(-p') i Res\left(\frac{-k^*(\gamma)}{k - k^*(\gamma)} e^{-ikp'}, k = k^*(\gamma)\right) \\
&= -\theta(-p') i k^*(\gamma) e^{-ik^*(\gamma)p'}
\end{aligned} \tag{4.82}$$

Thus, the generating function is,

$$G(\gamma) = \int \mathrm{d}p' I(\gamma, p') F(p') \tag{4.83}$$

We now transform back in γ to get an expression for the FCS. In particular we have,

$$\begin{aligned} P(\lambda) &= \int \frac{\mathrm{d}\gamma}{2\pi} G(\gamma) e^{-i\gamma\lambda} \\ &= \int \frac{\mathrm{d}\gamma}{2\pi} \mathrm{d}p' I(\gamma, p') F(p') e^{-i\gamma\lambda} \end{aligned} \tag{4.84}$$

Exchanging the order of integrals we compute,

$$\int \frac{\mathrm{d}\gamma}{2\pi} I(\gamma, p') e^{-i\gamma\lambda} \tag{4.85}$$

This integral is also performed by the usual combination of residues and Jordan's lemma. This time $\lambda > 0$ by construction so we must choose the semicircle in the lower half plane. The function $I(\gamma, p')$ has an essential singularity due to the presence of the exponential. The residue is found after a power series expansion around $\gamma = -i\epsilon$,

$$\begin{aligned} I(\gamma, p') e^{-i\gamma\lambda} &= -\theta(-p') i k^*(\gamma) e^{-ik^*(\gamma)p'} e^{-i\gamma\lambda} \\ &= \theta(-p') i \frac{8g}{m} (\gamma + i\epsilon)^{-1} e^{i\frac{8g}{m}(\gamma+i\epsilon)^{-1}p'} e^{-i(\gamma+i\epsilon)\lambda} e^{-\epsilon\lambda} \\ &= \theta(-p') i \frac{8g}{m} e^{-\epsilon\lambda} \sum_{n,m\geq 0} \frac{(i\frac{8g}{m}p')^n (-i\lambda)^m}{n!m!} (\gamma + i\epsilon)^{m-n-1} \end{aligned} \tag{4.86}$$

The coefficient of the first negative power gives the residue and is found when $m - n - 1 = -1$, that means $n = m$,

$$\begin{aligned} Res\left(I(\gamma, p') e^{-i\gamma\lambda}, \gamma = -i\epsilon\right) &= \theta(-p') i \frac{8g}{m} e^{-\epsilon\lambda} \sum_{n\geq 0} \frac{(\frac{8g}{m}p'\lambda)^n}{(n!)^2} \\ &= \theta(-p') i \frac{8g}{m} e^{-\epsilon\lambda} I_0\left(2\sqrt{\frac{8g}{m}p'\lambda}\right) \end{aligned} \tag{4.87}$$

where I_0 is the modified Bessel function of the first kind. Thus, taking into account a minus sign coming from the choice of the lower semicircle and another one coming from i^2 (one i from residue theorem and one from the above expression) we get,

$$\int \frac{\mathrm{d}\gamma}{2\pi} I(\gamma, p') e^{-i\gamma\lambda} = \theta(-p') \frac{8g}{m} e^{-\epsilon\lambda} I_0\left(2\sqrt{\frac{8g}{m}p'\lambda}\right) \tag{4.88}$$

and consequently, after letting $\epsilon \to 0^+$,

$$P(\lambda) = \frac{8g}{m} \int_{-\infty}^{+\infty} \mathrm{d}p' \theta(-p') I_0\left(2\sqrt{\frac{8g}{m}p'\lambda}\right) F(p') \tag{4.89}$$

The modified Bessel function $I_0(\sqrt{x})$ oscillates for $x \leq 0$. The asymptotic behavior of the modified Bessel functions is [87, p. 375-377],

$$I_0(x) = \frac{e^x}{\sqrt{2\pi x}}\left(1 + \frac{1}{8x} + \dots\right) \quad |x| \to +\infty \tag{4.90}$$

$$I_0(x) \sim 1 \quad x \to 0 \tag{4.91}$$

These expressions show that the integral is finite and that $P(\lambda) \to 0$ for $\lambda \to +\infty$ while it goes to a constant for $\lambda \to 0$. The theta function $\theta(-p')$ in the integral expression for the full counting statistics should ensure that this probability distribution is non-negative but a direct proof at this point is too hard. The interesting equation is (4.89). The only missing ingredient is the auxiliary function F. The Negro-Smirnov formula allows to compute it for only positive values of $p = q^2$ through the *r.h.s* of (4.70). The evaluation of (4.89) requires the knowledge for negative values of p: we face the problem of an analytic continuation.

4.4.3 An analytic continuation

Here we attempt to perform the aformentioned analytic continuation trying a replacement $q \to iy$ and study the consequences. The function ζ appearing in the *r.h.s.* of (4.70) satisfies the linear integral equation,

$$\zeta_q(p) = 1 + 4g\mathcal{P}\int \frac{\mathrm{d}\lambda}{2\pi}\frac{1}{p-\lambda}(2q - m\partial_\lambda)(n(\lambda)\zeta_q(\lambda)) \tag{4.92}$$

We define,

$$s_y(p) = n(p)\zeta_{iy}(p) \tag{4.93}$$

This quantity obeys,

$$n^{-1}(p)s_y(p) = 1 + 4g\mathcal{P}\int \frac{\mathrm{d}\lambda}{2\pi}\frac{1}{p-\lambda}(2iy - m\partial_\lambda)s_y(\lambda) \tag{4.94}$$

Taking the complex conjugate we see,

$$s_y = \bar{s}_{-y} \tag{4.95}$$

a fact that will be useful in a moment. The effect of $q \to iy$ on (4.70) is,

$$F(-y^2) = \exp\left(-\frac{8g}{m\pi}\int_0^y \mathrm{d}\tilde{y}\,\tilde{y}\int \mathrm{d}p s_{\tilde{y}}(p)\right) \tag{4.96}$$

We note a possible ambiguity for positive and negative values of y. Indeed, transforming $y \mapsto -y$ we find,

$$F(-y^2) = \exp\left(-\frac{8g}{m\pi}\int_0^y \mathrm{d}\tilde{y}\tilde{y}\int \mathrm{d}p\,\bar{s}_{\tilde{y}}(p)\right) \tag{4.97}$$

Of course, F should be real, as we are going to see. To study the integral equation we use *Dirac Braket* notation. With the definitions,

$$\Omega_{p,\lambda}(y) = 4g\mathcal{P}\frac{1}{2\pi}\frac{1}{p-\lambda}(2iy - m\partial_\lambda) \tag{4.98}$$

$$\Theta_{p,\lambda} = \delta(p-\lambda)n(\lambda) \tag{4.99}$$

Defining also the vector being constantly 1 as $|1\rangle$ and $|s_y\rangle$ the vector representing $s_y(p)$ we can write (4.94),

$$T_{p,\lambda}(y)\,|s_y\rangle = (\Theta^{-1}_{p,\lambda} - \Omega_{p,\lambda}(p))\,|s_y\rangle = |1\rangle \tag{4.100}$$

The linear operator T, being the sum of two self-adjoint operators, is self-adjoint. This can be seen going in Fourier space,

$$\tilde{\Omega}_{q,k}(y) = 2g\,\mathrm{sgn}(k)(y + mk)\delta(q-k) \tag{4.101}$$

Θ^{-1} is real by construction because n is real so that $T = T^\dagger$. Thus, its spectrum is real and its eigenvectores can be chosen to be orthonormal, $\langle n, y|m, y\rangle = \delta_{nm}$. Also, we order eigenvectors increasingly from the one for which the corresponding eigenvalue has the smallest norm (in $\mathbb{C}$) *i.e.* if $n < m$ then $|\langle n, y|n, y\rangle| < |\langle m, y|m, y\rangle|$. The spectral data will parametrically depend on y. In general the spectrum will be the union of a continuous and a discrete part, but we keep using a discrete notation, since we ultimately will discretize the operator and get a purely discrete spectrum. We stress that the operator T can have zero eigenvalues for some value of $y \in \mathbb{R}$. Indeed, we numerically verified this fact. Zero eigenvalues hide subtleties in the analytic continuation, as we now further comment. We define,

$$T_{p,\lambda}(y)\,|n, y\rangle = \mu_n(y)\,|n, y\rangle \tag{4.102}$$

The solution to (4.94) is formally written as,

$$|s_y\rangle = T^{-1}_{p,\lambda}(y)\,|1\rangle = \sum_n \frac{1}{\mu_n(y)}\,\langle n, y|1\rangle\,|n, y\rangle \tag{4.103}$$

Now, note that,

$$\langle 1|s_y\rangle = \int \mathrm{d}p\,\langle 1|p\rangle\,\langle p|s_y\rangle = \int \mathrm{d}p s_y(p) = \sum_n \frac{1}{\mu_n(y)}|\,\langle 1|n, y\rangle\,|^2 \tag{4.104}$$

Equation (4.104) says that the p-integral of $s_y(p)$ is real $\forall y$, showing that F is a real function. Of course, varying y eigenvalues vary too. Some of them can pass through zero causing singularities in (4.103). These singularities must be properly interpreted to make sense of the analytic continuation. To see if this interpretation is correct we regularize "by hand" the denominator of (4.103) as,

$$\int \mathrm{d}p s_y(p) = \lim_{\epsilon\to 0^+}\sum_n \frac{1}{\mu_n(y) \pm i\epsilon}|\,\langle n, y|1\rangle\,|^2 \tag{4.105}$$

Now, a function defined as,

$$g(y) = \exp\left(\lim_{\epsilon\to 0^+}\int_0^y \mathrm{d}x \frac{A}{B(x-y_0)\pm i\epsilon}\right) \tag{4.106}$$

with $A, B \in \mathbb{R}$ constants. Using (3.121),

$$\lim_{\epsilon\to 0^+}\int_0^y \mathrm{d}x \frac{A}{B(x-y_0)\pm i\epsilon} = \frac{A}{B}\log|y-y_0| - \frac{A}{B}\log|y_0| \mp i\frac{A}{|B|}\pi\theta(y-y_0) \tag{4.107}$$

where θ is the step function. Thus,

$$g(y) = \left|\frac{y-y_0}{y_0}\right| \exp\left(\mp i\frac{A}{|B|}\pi\theta(y-y_0)\right) \tag{4.108}$$

Since $e^{\pm i\pi\theta(x)} = \mathrm{sgn}(x)$ **if** it is true that $A/B = 1$ then,

$$g(y) = \frac{y_0 - y}{y_0} \tag{4.109}$$

which is real and analytic. In view of this simple observation we postulate (and numerically verify) the following Ansatz,

Ansatz 1. *Let y^* and n^* such that $\mu_{n^*}(y^*) = 0$, then*

$$\frac{-8g}{m\pi}|\langle n^*, y^*|1\rangle|^2 \frac{1}{\partial_y \mu_{n^*}(y)|_{y^*}} = 1 \tag{4.110}$$

If the Ansatz is correct we are free to replace $q \to iy$ in (4.55) and to obtain a valid analytic continuation to negative values of p interpreting zeros of F as singularities of $s_y(p)$. We also note that, this regularization procedure must be independent by the filling n. In order to reassure the reader about the validity of our Ansatz, we anticipate a few results presented in the next Chapter in Fig. 4.1. We analyze the analytic continuation employing physical fillings n, numerically computed in an actual quench protocol (see Chapter 2). In the Figure we plot the eigenvalue $\mu(y)$ of (4.103) of minimal norm as a function of y. Clearly, the function has isolated zeros, which lead to isolated zeros of F. In the same Figure we numrically verify the Ansatz and find excellent agreement.

Now that we have verified that our Ansatz is correct we can describe how, in practice, one can compute the function F (4.55). To isolate singularities in the integral equation (4.94) we write,

$$|s_y\rangle = |s_y^r\rangle + \left(\mu_0^{-1}(y) - \xi^{-1}\right)|0,y\rangle\langle 0,y|1\rangle \tag{4.111}$$

where we defined the regular part of $|s_y\rangle$ as,

$$|s_y^r\rangle \equiv \sum_n \left(\frac{1-\delta_{n,0}}{\mu_n(y)} + \delta_{n,0}\xi^{-1}\right)|n,y\rangle\langle n,y|1\rangle \tag{4.112}$$

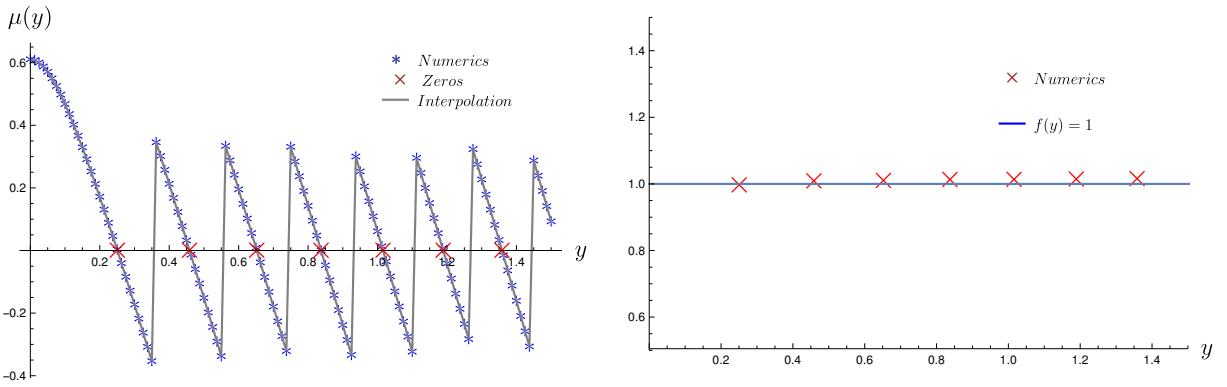

FIGURE 4.1: (Right) Numerical verification of the Ansatz (4.110). (Left) Zeros of the the eigenvalue with smallest norm as function of y. Steepest parts of the curve are actually jumps. We used 1^{st} order interpolation in order not to smooth these jumps. Here we used a physical filling with $g = 1$ and $m = 1/2$ but the Ansatz is independent on these choices.

This satisfies the equation,

$$(T_{p,\lambda}(p) + (\xi - \mu_0(y))\,|0,1\rangle\,\langle 0,1|)\,|s_y^r\rangle = |1\rangle \tag{4.113}$$

as can be seen by direct substitution. This is nothing else that a projection on the subspace ortogonal to $|0,y\rangle$ keeping explicint a regularizator ξ. Now, the generating function (4.55) in our notation is written as,

$$F(-y^2) = \exp\left[-\frac{8g}{m\pi}\int_0^y \mathrm{d}\tilde{y}\Big\{\tilde{y}\,\langle 1|s_{\tilde{y}}^r\rangle + \tilde{y}(\mu_0^{-1}(y) - \xi^{-1})|\,\langle 0,y|1\rangle\,|^2\Big\}\right] \tag{4.114}$$

Define the auxiliary function encoding the singularities,

$$f(y) = \sum_{i=1}^{\infty} \frac{y_i|\,\langle 0,y_i|1\rangle\,|^2}{\partial_y\mu_0(y)|_{y_i}}\frac{1}{y - y_i} \tag{4.115}$$

We sum and subract the poles,

$$\begin{aligned} F(-y^2) = &\exp\left[-\frac{8g}{m\pi}\int_0^y \mathrm{d}\tilde{y}\Big\{\tilde{y}\,\langle 1|s_{\tilde{y}}^r\rangle + \tilde{y}(\mu_0^{-1}(y) - \xi^{-1})|\,\langle 0,y|1\rangle\,|^2 - f(\tilde{y})\Big\}\right] \\ &\times \exp\left[-\frac{8g}{m\pi}\int_0^y \mathrm{d}\tilde{y} f(\tilde{y})\right] \end{aligned} \tag{4.116}$$

In this way we identify a regular part and a singular part,

$$F^r(-y^2) = \exp\left[-\frac{8g}{m\pi}\int_0^y \mathrm{d}\tilde{y}\Big\{\tilde{y}\,\langle 1|s^r_{\tilde{y}}\rangle + \tilde{y}(\mu_0^{-1}(y) - \xi^{-1})|\,\langle 0,y|1\rangle\,|^2 - f(\tilde{y})\Big\}\right] \tag{4.117}$$

$$F^s(-y^2) = \exp\left[-\frac{8g}{m\pi}\int_0^y \mathrm{d}\tilde{y} f(\tilde{y})\right] \to \prod_i^\infty \frac{y_i - y}{y_i} \tag{4.118}$$

where in the second equation we have used the Ansatz (4.110) and, after introducing an explicit regulator $i\epsilon$, carried out the integration similar to (4.106) in the limit $\epsilon \to 0$. Thus, we have found the analytic extension of F to negativ values of its argument,

$$F(-y^2) = F^r(-y^2)\prod_i^\infty \frac{y_i - y}{y_i} \tag{4.119}$$

Chapter 5

Numerical Study of the Relaxation to Steady States

In this last Chapter we present the details of the numerical methods we have used to directly simulate classical quenches of the Non Linear Schrödinger equation and to compute the full counting statistics of the particel density. The theory we are dealing with is defined on a continuum so that a numerical implementation requires a discretization which, of course, introduces errors due to the finite lattice spacing Δx. Moreover, since we draw initial field configuarionts from statistical ensembles, fluctuations arise requiring an average over a large number of initial instances. The code to compute one point correlation functions is made by three parts: the generation of random initial conditions and their microscopic time evolution, the computation of the transfer matrix and the solution of TBA equations. In the last Section we present the numerical procedure to find the FCS.

5.1 Numerical Simulation of Microscopic Dynamics

5.1.1 Time Evolution of Field Equation

To study the relaxtion to the steady state we directly simulate the time evolution of the system on a computer. A very efficient and stable implementation has been used in [88] in the context of the Gross-Pitaevski equation and we use a similar method. The NLSE is just the homogeneous case of the latter. The idea of the implementation is that of the *trotterization* of the hamiltonian, alternating between real and Fourier space. To time evolve the initial condition we start by splitting the Hamiltonian in the following way,

$$H = H_1 + H_2 + H_3 \tag{5.1}$$

with,

$$H_1 = g|\psi|^2\psi \tag{5.2}$$
$$H_2 = -\partial_x^2 \tag{5.3}$$
$$H_3 = H_1 \tag{5.4}$$

and consider an infinitesimal time step Δt and lattice spacing Δx. The NLSE equation can be written as,

$$i\partial_t\psi = H\psi \tag{5.5}$$

with formal solution,

$$\psi(x, t+\Delta t) = e^{-iH\Delta t}\psi(x,t) \sim e^{-iH_1\Delta t}e^{-iH_2\Delta t}e^{-iH_3\Delta t}\psi(x,t) \tag{5.6}$$

This splitting is convenient since $H_{1,3}$ are diagonal in real space while H_2 is diagonal in Fourier space. On the lattice we have,

$$\psi(x,t) \mapsto \psi_{i,j} \tag{5.7}$$

In order to evolve according to H_2 we first go in Fourier space and then go back into real space. The trotterization scheme proceed in three steps, according to the Hamiltonian splitting. The first step is,

$$\psi_{i,j+1/3} = e^{-iH_1\Delta t}\psi_{i,j} \tag{5.8}$$

where the field in H_1 is approximated as $\psi_{i,j}$. The second is,

$$\psi_{i,j+2/3} = e^{-iH_2\Delta t}\psi_{i,j+1/3} \tag{5.9}$$

To evaluate tha r.h.s. we go in Fourier space. For N lattice points, the Fourier Transform is,

$$\psi_{s,j} = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}ks}\tilde{\psi}_{k,j} \tag{5.10}$$

and the solution to $i\partial_t\psi_{i,j} = H_2\psi_{i,j}$ is,

$$\tilde{\psi}_{k,j+2/3} = e^{-i\epsilon(k)\Delta t}\tilde{\psi}_{k,j+1/3} \tag{5.11}$$

where $\epsilon(k)$ is the dispersion relation arising from the laplacian. Its discretization is,

$$\partial_x^2\psi(x,t) \mapsto \frac{\psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j}}{(\Delta x)^2} \tag{5.12}$$

Thus, on the lattice we have,

$$\epsilon(k) = \frac{2}{(\Delta x)^2}\left(1 - \cos\left(\frac{2\pi k}{N}\right)\right) - \mu \tag{5.13}$$

where μ is the chemical potential. Going back into real space, we evolve according to H_3 we obtain the solution after a time step Δt,

$$\psi_{i,j+1} = e^{-iH_3\Delta t}\psi_{i,j+2/3} \tag{5.14}$$

This way efficiently solves the theory on the lattice. Better results are obtained substituting the dispersion relation on the lattice with its continuum counterpart. That is,

$$\epsilon(k) = \left(\frac{2\pi k}{N\Delta x}\right)^2 - \mu \tag{5.15}$$

Notice that the hopping terms in the discretized second derivative couple nearest neighbours sites on the spatial lattice while the dispersion relation on the continuum means coupling many the points on the lattice at once. With the new dispersion relation we cutoff $k_j = \frac{2\pi j}{N\Delta x}$ in the UV to a certain value Λ. Since, in principle, the theory is defined on the lattice and we use periodic boundary conditions the Brillouin zone is $[-\pi/2, \pi/2]$ so that we can reasonably choose the high-momenta cutoff as $\Lambda = \pi/\Delta x \sim 3/\Delta x$.

The choice of the infinitesimal time step deserves explanation: indeed to have a true infinitesimal time evolution we should have that the exponential terms in the trotterization are small enough. Thus it must hold,

$$k_{max}^2 \Delta t \ll 1 \implies \Delta t \ll \left(\frac{\Delta x}{\pi}\right)^2 \tag{5.16}$$

because the maximum momentum k_i is attained at $i = N/2$. This means very short time steps. Indeed, in most of our simulations we have takes,

$$\Delta t = 0.1 \left(\frac{\Delta x}{\pi}\right)^2 \tag{5.17}$$

Note that if $\Delta x = 0.05$, $\Delta t \sim 2.53 \times 10^{-5}$ resulting in thousands of iterations already for a total evolution time of one unit.

5.1.2 Average and errors estimate

The classical version of eq. (4.26) predicts steady states of correlation functions. Such stationary values are checked numerically solving the NLSE a certain number of instances and measuring the quantities $O_K = \langle(\psi(x,t)^\dagger)^K \psi(x,t)^K\rangle$ as functions of time and eventually smoothing out space fluctuations averaging over them. In practice we solve the NLSE with an initial random configuration according to some values of parameters in the hamiltonian and average over such number of configurations. In particular, initial field configurations are drawn by thermal states as well as GGE's. In the case we consider it is easy to generate the configurations since initial distributions are gaussians with variance given by the lattice correlator,

$$\langle \bar{\psi}_k \psi_q \rangle = \frac{\delta_{k,q}}{\Delta x} f(k) \tag{5.18}$$

where $f(k) = 1/(\beta\epsilon(k))$ for thermal states and $f(k) = e^{-\beta\epsilon(k)}$ for GGe's we consider. Here $\epsilon(k)$ is the dispersion relation (can be chosen to be that on the lattice or on the continuum). Since the field is complex to generate $\psi(x,0)$ we consider the field in Fourier space $\tilde{\psi}(k,0) = re^{i\alpha}$. We draw from a gaussian distribution $\mathcal{N}(0, \langle \bar{\psi}_k \psi_q \rangle)$ the modulus r and from a uniform distribution the phase $\alpha \in [0, 2\pi]$. Then we go back in real space and get $psi(x,0)$. A generic observable in computed as,

$$\langle O \rangle = \frac{1}{N_{\mathcal{C}}} \sum_{\mathcal{C}} O[\mathcal{C}] \tag{5.19}$$

where $\mathcal{C}$ is a configuration of the field and $N_{\mathcal{C}}$ is the number of such configurations. The error estimate is the standard deviation computed in the usual way. As a matter of fact, the computational resources required to get precise results are huge, since we need to follow the time evolution of observables for $N_{\mathcal{C}} \gg 1$. Luckly, we had at our disposal the University of Milan computer cluster and the SISSA cluster. This allowed us to reach good precision. What we did is to split the average procedure in two steps. Suppose we have a N_b cores and we want to average over $N_{\mathcal{C}}$ configurations. The strategy is to run

a simluation which averages over $N_i = N_{\mathcal{C}}/N_b$ for each core. After that, we average over the number of cores. At this point the error estimate requires attention. Let us call $\langle \circ \rangle = \frac{1}{N_b}\sum_{j=1}^{N_b} \circ_j$, $\bar{\circ} = \frac{1}{N_i}\sum_{j=1}^{N_i} \circ_j$ and let x be a random variable. We are interested in computed the error estimate for,

$$\langle \bar{x} \rangle = \langle \frac{1}{N_i} \sum_{j=1}^{N_i} x_j \rangle \tag{5.20}$$

It is easy to show that [citation] for a linear function of N uncorrelated random variables $\underline{x}$, $f(\underline{x}) = \sum_{i=1}^{N} a_i x_i$ the error propagation gives,

$$\sigma^2_{f(\underline{x})} = \sum_{i=1}^{N} a_i^2 \sigma_i^2 \tag{5.21}$$

being σ_i^2 the standard deviation of the random variable x_i. In our case $f(\underline{x}) = \bar{x}$. Then our error estimate for eq. 5.20 is,

$$\sigma^2_{\bar{x}} = \frac{1}{N_i} \sum_{j=1}^{N_i} \sigma^2_{x_j,b} \tag{5.22}$$

where the subscript b indicates that the variance is with respect to N_b values. Since the latter scales as $\frac{1}{N_b}$, the total error scales as $\frac{1}{\sqrt{N_b N i}} = \frac{1}{\sqrt{N_{\mathcal{C}}}}$ as we wanted. This way of proceed, despite conceptually very simple, allows to have good precision in a reasonable amount of time.

5.1.3 Recovering the continuum limit

Eq. (4.26) as well as eq. (4.52) predicts stationary values for a continuum theory. In the numerical solution of the NLSE we discretize the theory, naturally introducing a cutoff Δx. The continuum theory is recovered as $\Delta x \to 0^+$. To find the correct values of observables we need to perform an extrapolation. This can be done assuming that each measured observable is Taylor expandable as a function of the lattice spacing, the 0^{th} order of the expansion being the continuum limit. Therefore, studying various lattice spacings we can extrapolate the value of the observable in the continuum limit. Indeed, an observable f is writtend as,

$$f = f(t, \Delta x) \tag{5.23}$$

where t is time. Expanding in power series in the lattice spacing we have,

$$f(t, \Delta x) = f(t, \Delta x = 0) + \Delta x f'(t, \Delta x)|_{\Delta x = 0} + \dots \tag{5.24}$$

Simulating and measuring for different values of Δx allows to fit the intercept and recover the continuum limit as shown in Fig. 5.1.

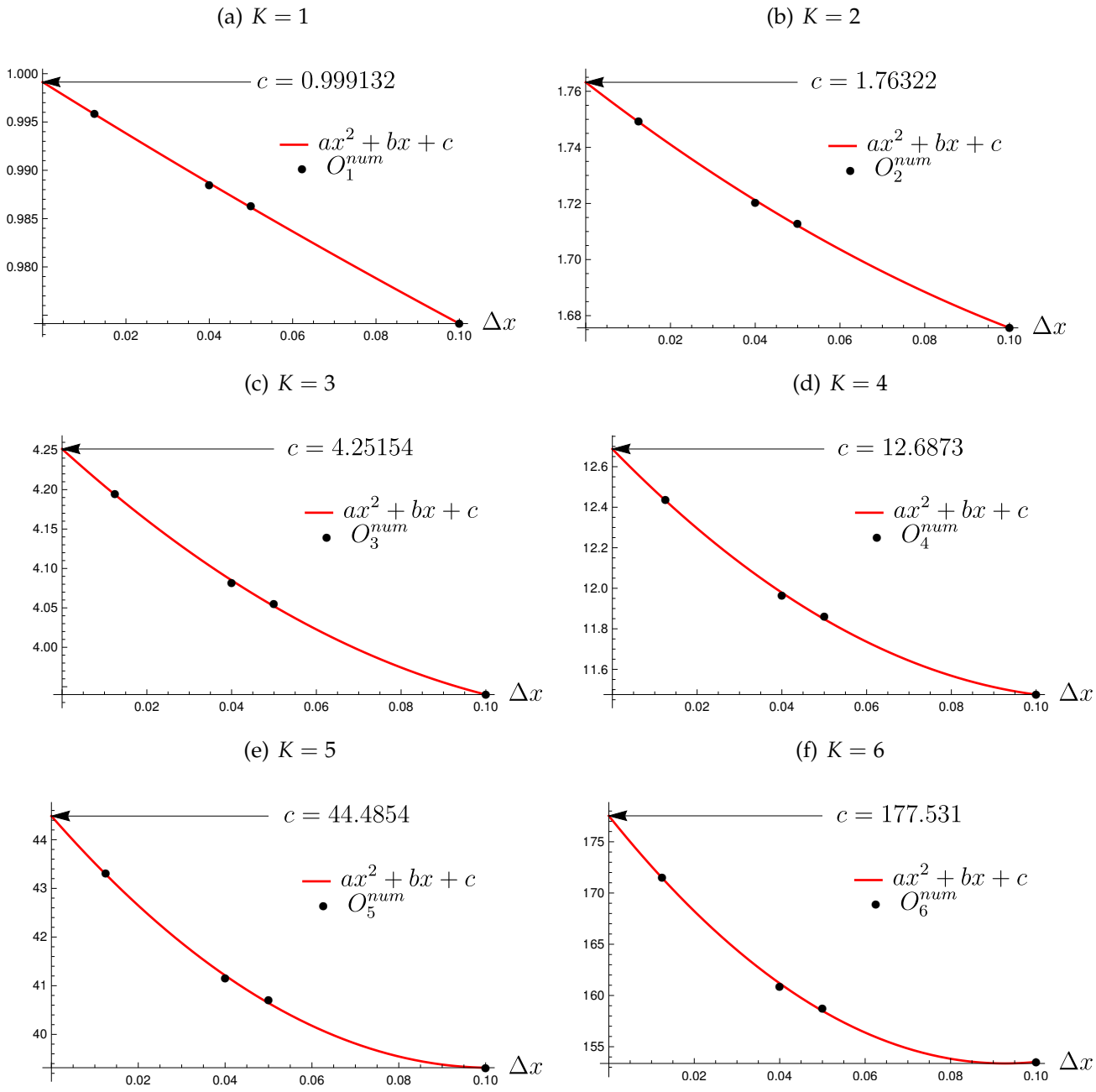

FIGURE 5.1: Extrapolation of $O_K = \langle(\bar{\psi}\psi)^K\rangle$ from direct numerical simulation for a quench on the thermal state in (5.18) with $g_i = 0$ to $g_f = 1$, $\beta = 0.4$, $L = 150$, $N_{\mathcal{C}} = 250 \times 10^3$ and unit density $D = 1$.

5.2 Numerical computation of Transfer Matrix

The numerical computation of the Transfer Matrix can be done in many ways. Our apporach will be the following: consider eq. (3.116),

$$\partial_x F = U_\lambda F \tag{5.25}$$

with,

$$U_\lambda = \begin{pmatrix} -\frac{i\lambda}{2} & \sqrt{g}\bar{\psi} \\ \sqrt{g}\psi & \frac{i\lambda}{2} \end{pmatrix} \tag{5.26}$$

Integrating at first order in Δx we find,

$$F(x+\Delta x,t) = W(\Delta x;x,t)F(x,t) \tag{5.27}$$

where,

$$W(\Delta x;x,t) \doteq \exp(\Delta x U_\lambda) = \begin{pmatrix} \cosh(\Delta x\mu) - \frac{i\lambda}{2\mu}\sinh(\Delta x\mu) & \frac{\sqrt{g}\bar{\psi}\sinh(\Delta x\mu)}{\mu} \\ \frac{\sqrt{g}\psi\sinh(\Delta x\mu)}{\mu} & \cosh(\Delta x\mu) + \frac{i\lambda}{2\mu}\sinh(\Delta x\mu) \end{pmatrix} \tag{5.28}$$

$$\mu \doteq \frac{1}{2}\sqrt{4g|\psi|^2 - \lambda^2} \tag{5.29}$$

If the field is supported on a finite interval $I = [a,b]$, define $\Delta x = \frac{b-a}{N}$ for some large N,

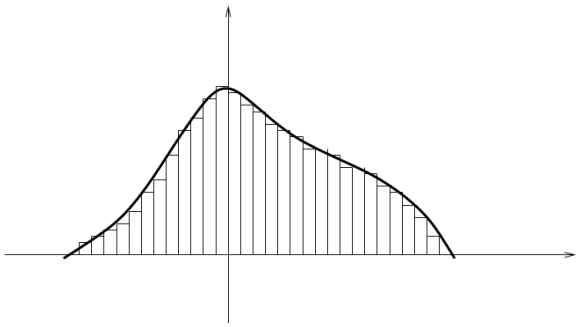

FIGURE 5.2: Cartoon of the discretization of the potential used to compute the transfer matrix (5.30).

$x_n = a + n\Delta x$, and approximate as,

$$T_\lambda(t) \simeq \prod_{n=0}^{N} W(\Delta x; a+n\Delta x, t) \tag{5.30}$$

***Remark* 2.** *In order to study the thermodynamic limit one should work with large values of L. This is problematic, given the presence of hyperbolic functions in eq.* (5.28)*. To avoid numerical instability, the strategy is the following. Our last aim is to extract eq.* (3.145)*. The pseudo-code is,*

$$\begin{aligned} v_1 &\leftarrow W(\Delta x;0,t)\begin{pmatrix}1\\0\end{pmatrix} \\ w_1 &\leftarrow \frac{v_1}{||v_1||} \\ R &\leftarrow \log||v_1|| \end{aligned}$$

Now we compute,

$$\begin{aligned} w_i &\leftarrow W(\Delta x; i\Delta x, t) w_{i-1} \\ w_i &\leftarrow \frac{w_i}{||w_i||} \\ R &\leftarrow R + \log ||w_i|| \end{aligned}$$

At the end we find,

$$v_N = w_N e^R \implies \frac{\log |a|}{L} = R/L + \log \left| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot w_N \right| / L$$

because,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot v_N = \begin{pmatrix} 1 & 0 \end{pmatrix} \prod_{n=0}^{N} W(\Delta x; n\Delta x, t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} T_\lambda(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a$$

The same averaging over random configuration is done to compute the average root density.

5.2.1 A Check in the Fully Homogeneous Case

In the case of homogeneous field (constant in space) it is possible to explicitely compute the form of the root density in the large L limit and to solve the NLSE. Indeed, eq. (5.30) becomes,

$$T_\lambda(t) = W(N\Delta x, t) = W(L, t) \tag{5.31}$$

where W this time does not depend on the coordinate x and $N\Delta x = L$ is the volume of the system. Thus,

$$\begin{aligned} \rho(\lambda) &= \frac{1}{g\pi} \lim_{L\to\infty} \frac{1}{2L} \log \left(\cosh^2(L\mu) + \frac{\lambda^2}{4\mu^2} \sinh^2(L\mu) \right) \\ &= \frac{1}{g\pi} \lim_{L\to\infty} \frac{1}{2L} \log \left(\frac{e^{2L\mu}}{4} \left(1 + \frac{\lambda^2}{4\mu^2} \right) \right) = \frac{1}{g\pi} \mu = \frac{1}{2g\pi} \sqrt{4g|\psi|^2 - \lambda^2} \end{aligned} \tag{5.32}$$

The *semi-circle law*, see Fig. 5.32 is traditionally written as,

$$p(x) = \frac{2}{\pi R^2} \sqrt{R^2 - x^2} \tag{5.33}$$

and it is supported in $[-R, R]$. Its moments are given by,

$$\langle x^{2n} \rangle = \left(\frac{R}{2} \right)^{2n} C_n \tag{5.34}$$

where,

$$C_n = \frac{1}{n+1} \binom{2n}{n} \tag{5.35}$$

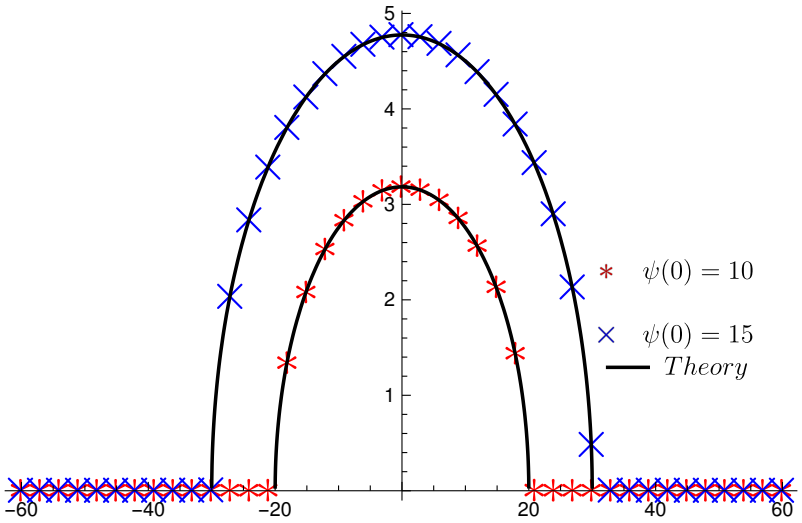

FIGURE 5.3: Numerical root density against theoretical prediction in the case of homogeneous field. In this case we have taken $L = 150$, $\Delta x = 0.001$ and $\Delta\lambda = 0.15$.

are the Catalan numbers. The distribution is not normalized as a probability density but to the particle density,

$$\int_{\mathbb{R}} \rho(\lambda) d\lambda = |\psi|^2 \tag{5.36}$$

as it should be for a root density. We recognize that here $R^2 = 4g|\psi|^2$ so that,

$$\int_{\mathbb{R}} \rho(\lambda)\lambda^{2n} d\lambda = |\psi|^2 \left(g|\psi|^2\right)^n C_n \tag{5.37}$$

In the homogeneous case, we can actually say more: the full solution can be explicitely computed. If $\partial_x \psi(x,t) = 0$, the NLSE reduces to,

$$i\partial_t \psi = 2g|\psi|^2\psi \tag{5.38}$$

Multiplying by $\bar{\psi}$ and subtracting the complex conjugate equation we get the integral of motion (we already knew that the number density was conserved in the inhomogeneous case),

$$\partial_t |\psi|^2 = 0 \tag{5.39}$$

so that the time evolution of the field is a pure phase. If $\psi(t_0) = \psi_0$, the full solution is,

$$\psi(t) = \psi_0 e^{-2ig|\psi_0|^2(t-t_0)} \tag{5.40}$$

It is interesting to note that expectation values of conserved quantities, which are obtained from the moments of the root density, depend only of the constant of motion $|\psi|^2$ and the coupling: as a consequence they are strictly positive.

5.3 Numerical solution of TBA equations

5.3.1 Fourier Approach

Since the equations (4.52) are linear a Fourier approach might be faster and more intuitive. Let us start by considering the linear operator defined by,

$$O_1[f](\lambda) = \mathcal{P}\int_{\mathbb{R}} \frac{d\mu}{2\pi}\frac{f(\mu)}{\lambda-\mu} = \frac{1}{2\pi}\mathcal{P}\left(\frac{1}{\lambda}\right)\star f \tag{5.41}$$

where the star indicates the convolution product in distribution space. In Fourier space[1] the convolution becomes a product,

$$\widetilde{O_1[f]}(k) = -\frac{i}{2}\,\mathrm{sgn}\,(k) f(k) \tag{5.42}$$

For the operator,

$$O_2[f](\lambda) = \mathcal{P}\int_{\mathbb{R}} \frac{d\mu}{2\pi}\frac{\partial_\mu f(\mu)}{\lambda-\mu} = \frac{1}{2\pi}\mathcal{P}\left(\frac{1}{\lambda}\right)\star \partial f \tag{5.43}$$

ans so,

$$\widetilde{O_2[f]}(k) = \frac{1}{2}k\,\mathrm{sgn}\,(k) f(k) \tag{5.44}$$

We now consider the same discretization we used in the direct approach, where $\lambda_i = \Delta i$ and $i \in \{-N+1, \ldots, N\}$ (the same for μ). Define the $2N \times 2N$ diagonal matrix

$$D_{jk} = \epsilon(\lambda_j)\delta_{jk} \tag{5.45}$$

and the vector

$$b_j = b(\lambda_j) \tag{5.46}$$

Also the diagonal matrices in Fourier space,

$$A_{jk} = -\frac{i}{2}\,\mathrm{sgn}\,(k_k)\delta_{jk} \quad I_{jk} = \frac{1}{2}k_j\,\mathrm{sgn}\,(k_j)\delta_{jk} \tag{5.47}$$

The matrix which implements the Fourier Transform is,

$$F_{js} = e^{-ik_s\lambda_j} \tag{5.48}$$

with $\lambda_j = \Delta j$ and $k_s = \frac{2\pi}{2N\Delta}s$. These definitions and discretizations allow us to write eq. (4.52) in the following way,

$$Db_n = F^{-1}(2Ap_nFb_{n-1} - 4mgIFb_n) \tag{5.49}$$

and the solution is easily computed,

$$b_n = 2p_nUb_{n-1} \quad U = [D + 4mgF^{-1}IF]^{-1}F^{-1}AF \quad n > 1 \tag{5.50}$$

[1] Only in this paragraph Fourier trnasform is defined as $f(x) = \frac{1}{2\pi}\int \mathrm{d}k e^{ikx}\tilde{f}(k)$.

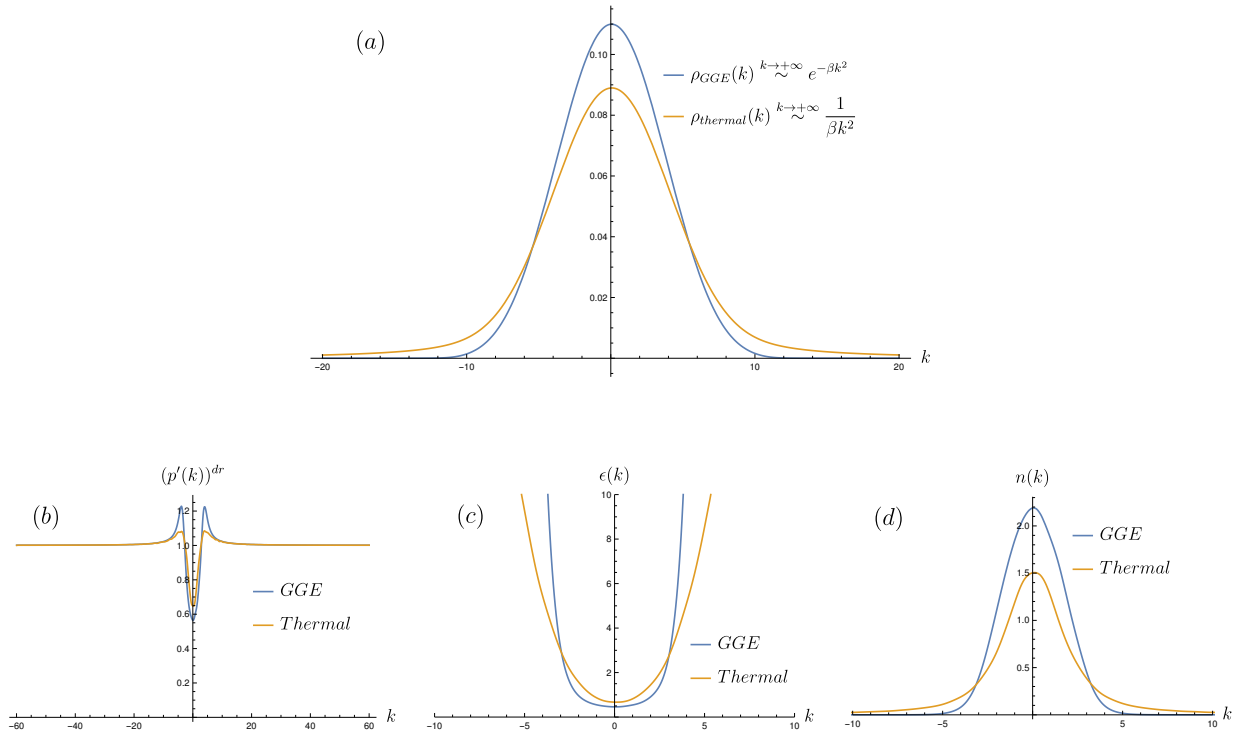

FIGURE 5.4: The plots show a comparison between different quantities involved in the computation of the one point functions for a quench from a GGE and thermal state. Parameters are $g_i = 0, g_f = 1, \beta = 0.4, L = 150$, $N_{\mathcal{C}} = 250 \times 10^3$, $D = 1$. (a) Numerical root density computed as in (5.30). (b) Numerical dressing of $p'(k)$. Note that without dressing this quantity should be constantly 1, see (2.150). The interaction causes non trivial deviation from this value. (c) Pseudoenergy (dressed energy), see (2.151). (d) Filling fraction, see (2.119)

.

For $n = 1$ we have,

$$b_1 = [D + 4mgF^{-1}IF]^{-1}v \quad v_i = 1 \quad \forall i \in \{1, \dots, 2N\} \tag{5.51}$$

Iteration gives b_n for every n. In Fug. 5.4 we show different quantities involved in the process of computing one point functions. The ultimate goal is to get the pseudoenergy which serve as input to compute b_n. In Fig. 5.5 we extrapolate one point functions obtained from the b_n, since there is a dependence in the lattice spacing due to (5.30).

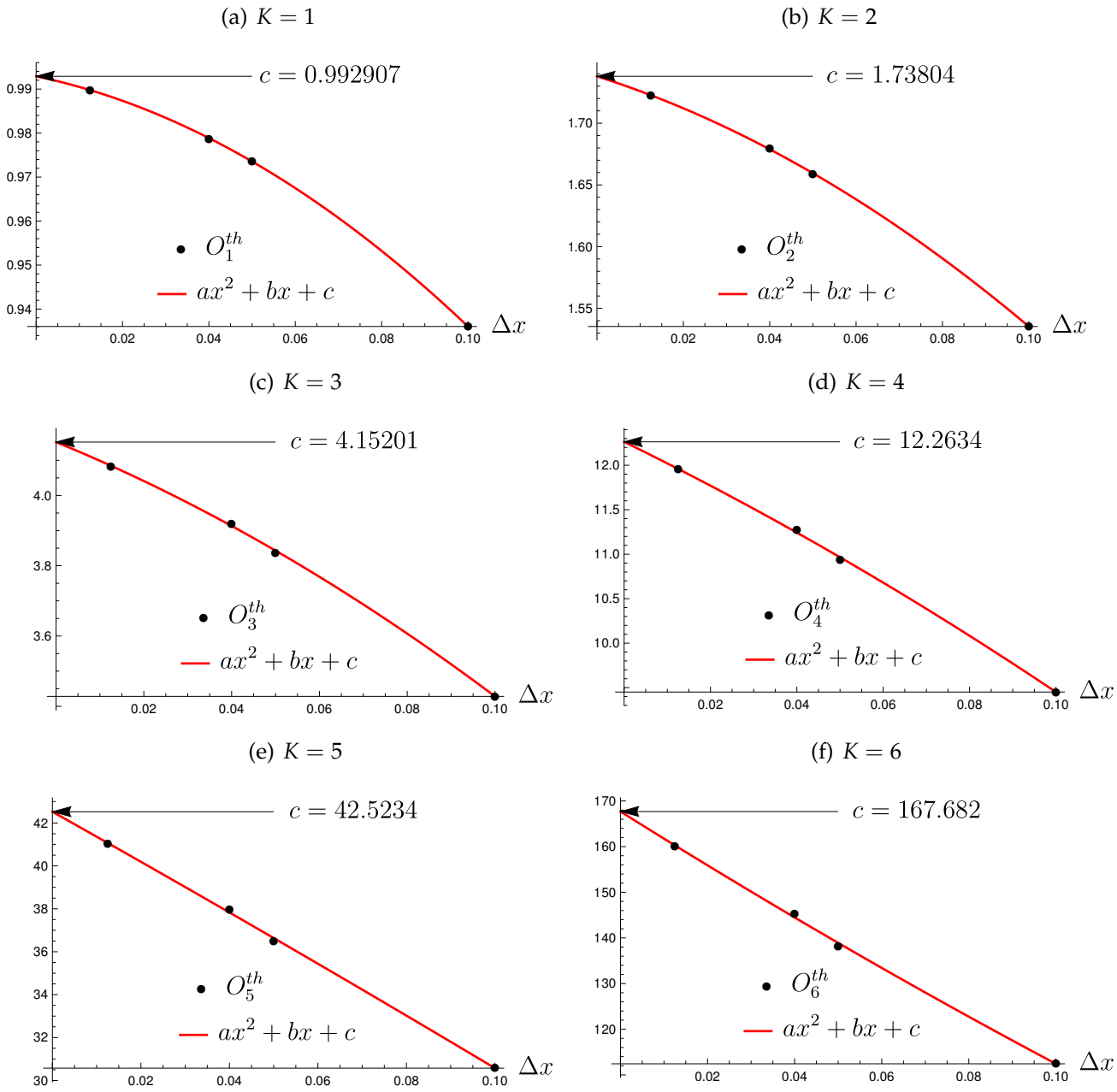

FIGURE 5.5: Extrapolation of O_K from the numerical solution of (4.52) for a quench form the thermal state in (5.18) with $g_i = 0$ to $g_f = 1$, $\beta = 0.4$, $L = 150$, $N_{\mathcal{C}} = 250 \times 10^3$ and unit density $D = 1$.

5.4 Comparison between Theory and Numerics

Fig. 5.6 shows the relaxation of one point functions after a quench from an initial GGE. Parameters are reported in the caption. There is clear convergence towards theoretical stationary values. Fig. 5.7 shows the relative error between theoretical prediction for steady state one point functions obtained by the solution of TBA equations and direct numerical simulation of the dynamics for a quench from a thermal state. We find very good agreement but as reported in Fig. 5.8 the higher the power of $(\bar{\psi}\psi)^K$ the higher the error. This expected since an error δ in $\bar{\psi}\psi$ explodes as we compute $(\bar{\psi}\psi)^K$. Despite we are able to qualitatively explain the error grow, a more careful analysis of different error sources may be needed for more complicated situations.

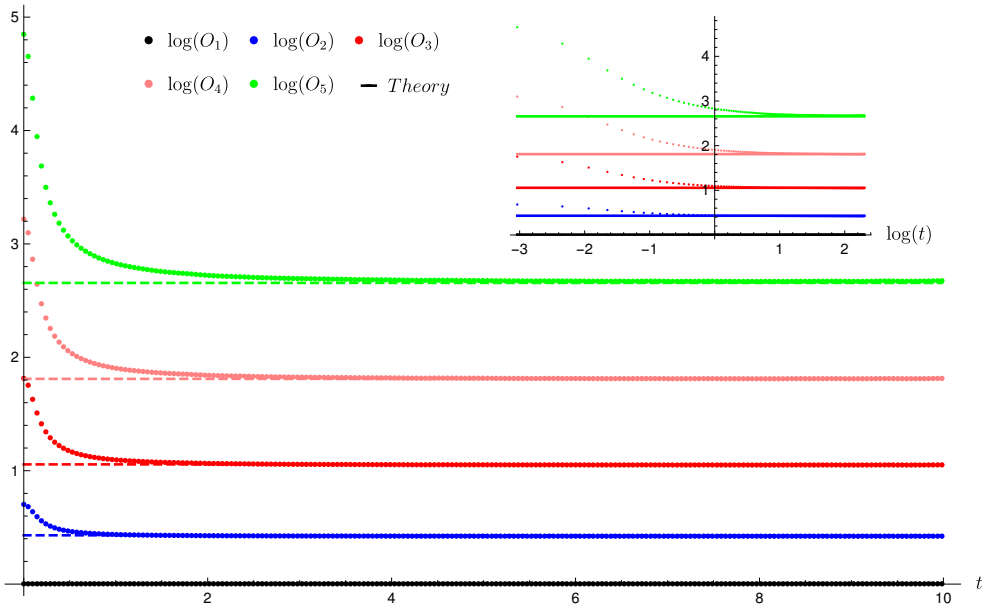

FIGURE 5.6: Dashed lines are theoretical prediction for steady state one point functions for a quench from the GGE initial state (2.47). Predictions are obtained by numerically solving (4.52). The input is the root density obtained numerically from the Inverse Scattering Method and numerically obtaiened as in (5.30). Parameters are $\beta = 0.4$, $L = 150$, $g_i = 0$, $g_j = 1$, $\Delta x = 0.05$, $D = 1$.

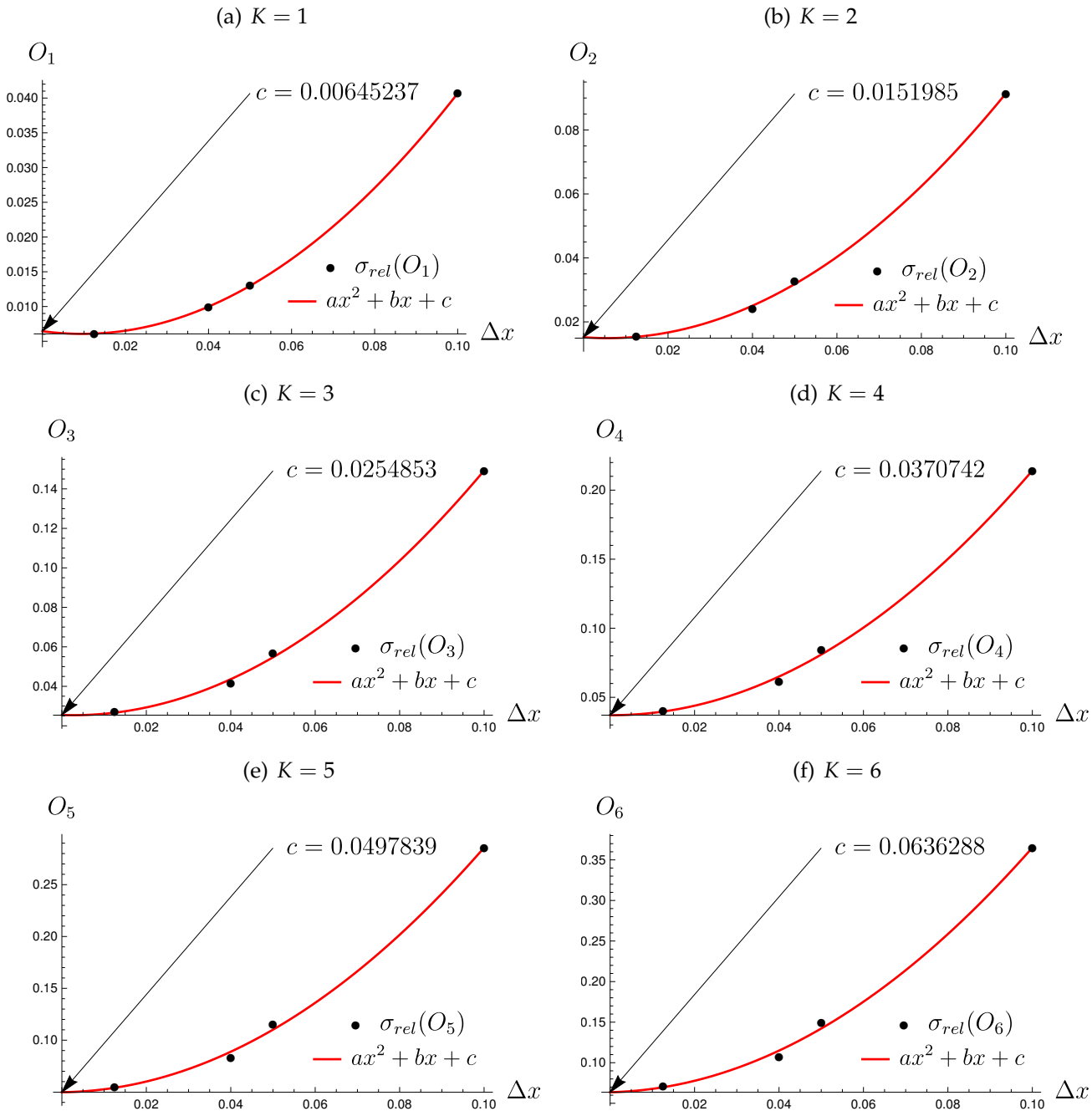

FIGURE 5.7: Relative error between theoretical predictions extrapolated as in Fig. 5.5 and microscopic simulation extrapolated as in Fig. 5.1. We find good agreement between theory and numerics.

5.5 Numerical Findings for the FCS

5.5.1 Microscopic Simulation

The direct numerical simulation of the steady state FCS is analogous to that of its moments. We have seen in Section 5.1 how to reach the steady state evolving the initial

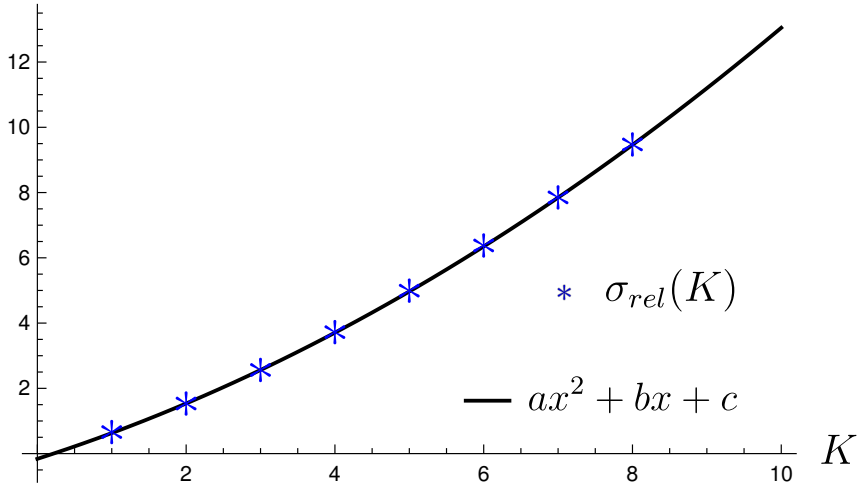

FIGURE 5.8: Growth of percentage error. An error δ in the density $\langle\bar{\psi}\psi\rangle$ blows up when computing $\langle(\bar{\psi}\psi)^K\rangle$.

random field configuration. In this case we define the function,

$$O(d,|\psi|^2) = \begin{cases} \frac{1}{\delta} & |\psi|^2 \in [d-\delta/2, d+\delta/2] \\ 0 & otherwive \end{cases} \tag{5.52}$$

We measure the function $O(d,|\psi|^2)$ at a certain time $T \gg 1$ for many random configurations and for sufficiently small δ. For $\delta \to 0^+$,

$$\langle O(d,|\psi|^2)\rangle \to P(\lambda) \tag{5.53}$$

We find that a good choice is already $\delta = 0.1$.

5.5.2 The generating function

In order to compute the function $F(p)$ for every p we need to solve the integral equation (4.94). We proceed discretizing the linear operator appearing on the *r.h.s.*. We have not work to do since the operators are the same of the previous section. Checking the Ansatz is very simple: after sampling the eigenvalue with minimum absolute value for some y, we find numerically its zeros, compute the corresponding eigenvector and apply formula (4.110), see Fig. 4.1. After, we compute the regular part (4.117). To do this we first need the function s_y^{reg}. This is computed by numerically solving (4.113). This function depends on two parameters, p and y so that here we show only the p-integral, see Fig. 5.12. Inserting the numerical result in (4.119), knowing the zeros we find the auxiliary function. A first check is to compute the FCS in a thermal state with $g = 0$. Since the initial distribution is gaussian, in this case it is straightforward to compute the FCS,

$$P(\lambda) = \frac{1}{D} e^{-\frac{\lambda}{D}} \tag{5.54}$$

where $D = \langle|\psi|^2\rangle$ is the density.

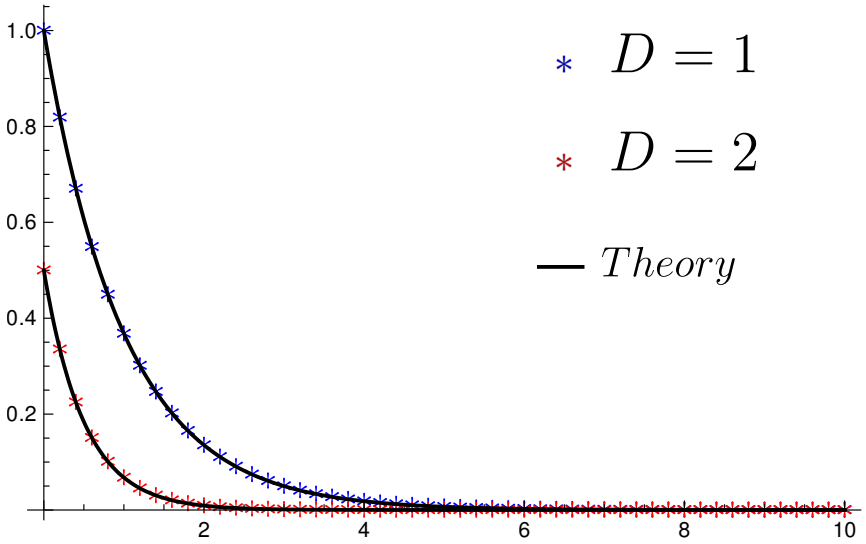

FIGURE 5.9: Agreement between theoretical prediction and numerically computed FCS in the free theory for different densities $N_{\mathcal{C}} = 10^4$.

In Fig. 5.10 we show the agreement for a quench from a GGE for two different values of the density while in FIg. 5.11 we show a quench from a thermal state.

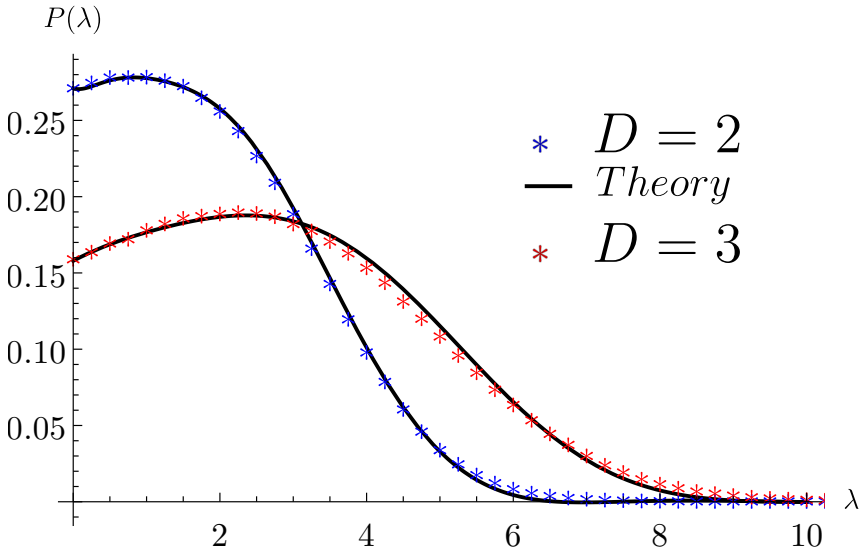

FIGURE 5.10: Agreement between theoretical prediction and numerically computed FCS for a quench from a GGE with $\beta = 0.4$, $g_i = 0$,$g_f = 1$, $N_{\mathcal{C}} = 10^4$.

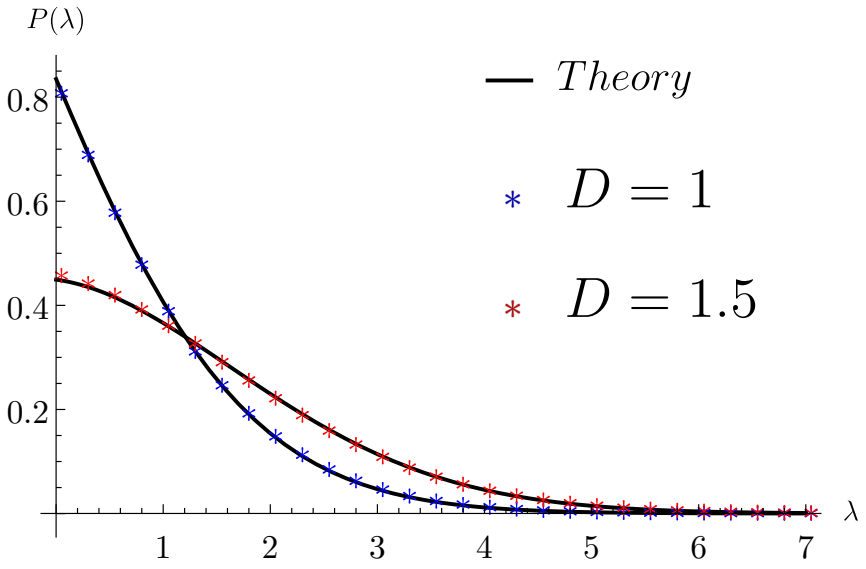

FIGURE 5.11: Agreement between theoretical prediction and numerically computed FCS for a quench from a thermal state with $\beta = 0.4$, $g_i = 0$,$g_f = 1$, $N_{\mathcal{C}} = 10^4$.

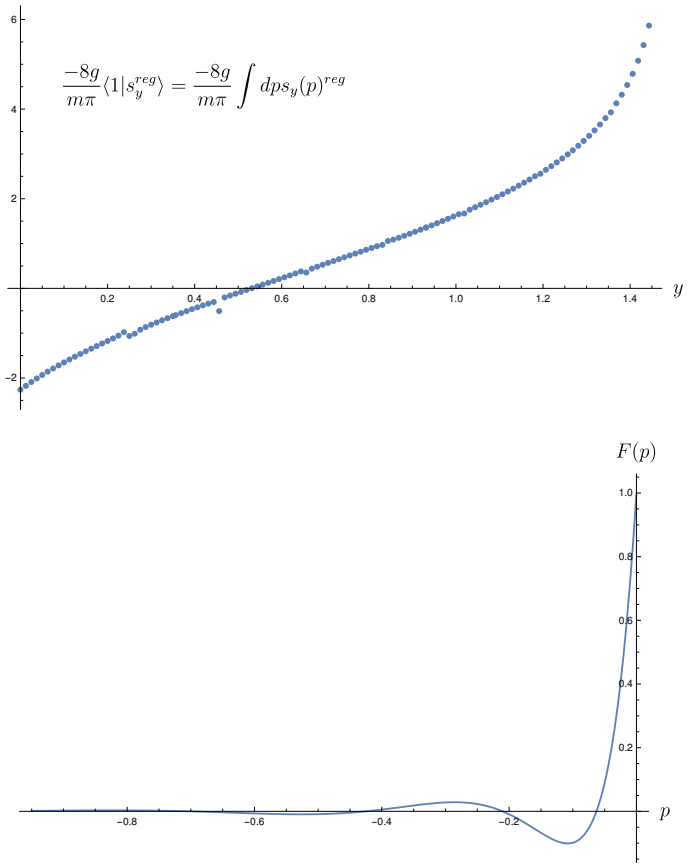

FIGURE 5.12: (Top)Numerical computation of the integrand in the exponent of (4.117). (Bottom) Function F in eq. (4.119).

Conclusions and Outlooks

In this thesis we have discussed the relaxation from out-of-equilibrium initial conditions in classical integrable field theories, in the thermodynamic limit and with extensive number of particles. To achieve this goal we have studied the classical analog of the homogeneous quantum quench: we have considered statistical ensembles of initial conditions, which are the classical counterparts of density matrices in the quantum setting, and followed their deterministic evolution towards the equilibrium state. We have seen that the fundamental quantity which determines the steady state, that is the equilibrium GGE, is the *root density*. This function can be taken from the Inverse Scattering Method, by which the Non Linear Schrödinger model hamiltonian is diagonalized. To verify the correcteness of this fact, we have taken the semiclassical limit of known one point functions of the quantum Lieb-Liniger which gives the one point functions of the Non Linear Schrödinger model and compared the exact predictions with microscopic numerical simulations finding excellent agreement. Moreover, exploiting the Negro-Smirnov formula and the correspondence between the classical Sinh-Gordon model, in the non relativistic limit, and the Non Linear Schrödinger model we were able to exactly compute the full counting statistics for the density of particles. Comparing with direct numerical simulations, also in this case, we found excellent agreement for a a wide range of initial conditions. Our method is, in principle, applicable to every integrable field theory. Once the initial condition is given, the root density is fully determined and its numerical computation is straghtforward.
In the future we aim to test our techniques on other classical integrable field theories. Also, the initial states we considered here are only a selected class of all the possibilities. We could consider interacting-to-interacting or interacting-to-non-interacting quenches. Inhomogenities and weakly integrable-breaking interactions are undergoing intensive research, with great focus on the quantum world. However, the same questions can be posed in the classical realm as well, where the combination of powerful analytical methods and refined numerical techniques give physicists an unmissable chance to better understanding such a difficult problem.

Appendices

Appendix A

Scattering Theory

A.1 The Schrödinger Equation

The idea behind scattering theory is simple: in order to test the physical properties of the system at hand we probe it with a projectile. The analysis of how something is bounched off let us recover information about the target. Here we want to discuss analytic properties of scattering solutions and introduce the S-matrix. As a simple example, we consider a one dimensional quantum system described by the Hamiltonian,

$$H = \frac{p^2}{2m} + V(x) \tag{A.1}$$

where the potential is supposed to vanish at infinity. The Schröringer equation for this system is,

$$\frac{-\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x) \tag{A.2}$$

We have to supplement the above equation with appropriate boundary conditions to model the physical situation we mentioned above. The problem is to determine the *spectrum*, that is the allowed values of the energy E and the eigenfunctions ψ associated to the eigenvalue E. It is well known tha there are two different kinds of solutions,

- *Bound states* for which the wave function decays exponentially fast $\psi \sim e^{-\chi|x|}$ as $x \to \pm\infty$ with negative energy $E = \frac{-\hbar^2\chi^2}{2m}$.
- *Scattering states* for which the wave function is plane wave $\psi \sim e^{ikx}$ as $x \to \pm\infty$ with positive energy $E = \frac{\hbar^2k^2}{2m}$.

Consequently the spectrum will consists of two parts, one continuous, for scattering states, and one discrete, for bound states. This nomenclature will recurr many times for the rest of the thesis. Every time a solution to an equation displays typical decay of a bound state will be called that way and the same for scattering states. To study anlytical properties of scattering it is better to work in the complex energy plane. For this puspose set for simplicity $\hbar = 1$ and $m = 1/2$. The energy is $E(k) = k^2$. Real k consitutes the continuous spectrum while imaginary k represents the bound states. The Schrödinger equation is linear and its solutions lie in a two dimensional vector space G_k. Superposing two basis vectors we can construct all solutions. Consider two basis specified by the boundary conditions at $x \to +\infty$,

$$\psi_1(x,k) = e^{-ikx} + o(1) \tag{A.3}$$

$$\psi_2(x,k) = e^{ikx} + o(1) \tag{A.4}$$

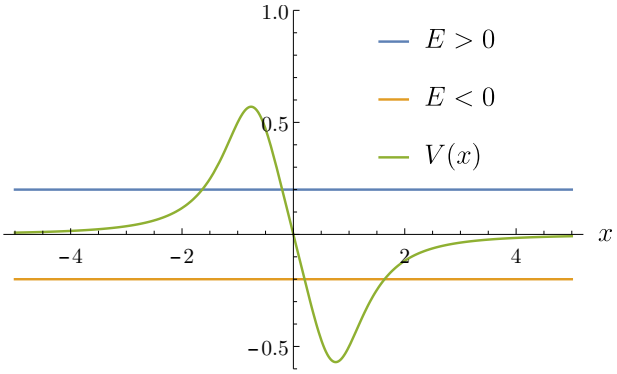

FIGURE A.1: An example of potential supporting scattering and bound states.

and at $x \to -\infty$,

$$\phi_1(x,k) = e^{-ikx} + o(1) \tag{A.5}$$

$$\phi_2(x,k) = e^{ikx} + o(1) \tag{A.6}$$

Since the potential is real for real k if $\psi(x,k)$ is a solution $\bar{\psi}(x,k)$ is a solution too. Thus,

$$\psi_1(x,k) = \bar{\psi}_2(x,k) \quad , \quad \phi_1(x,k) = \bar{\phi}_2(x,k) \tag{A.7}$$

In addition, it is evident that by construction,

$$\psi_1(x,k) = \psi_2(x,-k) \quad , \quad \phi_1(x,k) = \phi_2(x,-k) \tag{A.8}$$

These two basis can be transformed one into another by means of a linear transformation,

$$\phi_i(x,k) = \sum_{j=1,2} T_{ij}(k)\psi_j(x,k) \tag{A.9}$$

The matrix T is called *transition matrix* and encondes all the information about the scattering problem. By (A.7),

$$T(k) = \begin{pmatrix} a(k) & b(k) \\ \bar{b}(k) & \bar{a}(k) \end{pmatrix} \tag{A.10}$$

Since solutions are related by complex conjugation (A.7) we omit subscripts. We can write,

$$\phi(x,k) = a(k)\psi(x,k) + b(k)\bar{\psi}(x,k) \tag{A.11}$$

Using the Schrödinger equation, it is easy to show that the Wronskian $W(g_1,g_2) = g_1g' - g_2g_1'$ does not depend on x. Also, it is immediate to see $W(\phi,\bar{\phi}) = W(\psi,\bar{\psi}) = 2ik$. Together with (A.11) it gives the important relation,

$$\det T(k) = |a(k)|^2 - |b(k)|^2 = 1 \tag{A.12}$$

Let us know give a physical interpetation to the transition matrix. As $x \to +\infty$ we have,

$$\frac{\phi(x,k)}{a(k)} = e^{-ikx} + \frac{b(k)}{a(k)}e^{ikx} + o(1) \tag{A.13}$$

that is an incident wave e^{-ikx} from the right and a reflected wave $\frac{b(k)}{a(k)}e^{ikx}$. As $x \to -\infty$,

$$\frac{\phi(x,k)}{a(k)} = \frac{e^{-ikx}}{a(k)} + o(1) \tag{A.14}$$

which is a transmitted wave. This means that $t = a^{-1}$ and $r = ba^{-1}$ are the transmission and reflection coeffiecients respectively. In term of these functions,

$$|r(k)|^2 + |(t(k)|^2 = 1 \tag{A.15}$$

The coefficients a, b, and so t, r, satisfy very important analytical properties as we now show. Before we turn the attention to the discrete spectrum of the Schrödinger equation. Bound states solutions satisfy,

$$-\psi_n''(x) + V(x)\psi_n(x) = -\chi_n^2\psi_n(x) \tag{A.16}$$

where the have put $k = i\chi$. Solutions are specified again by the asymptotic behaviors,

$$\psi_n(x) = c_{n,\pm}e^{\mp\chi_n x} + O(1) \quad x \to \pm\infty \tag{A.17}$$

Thanks to the normalization condition it is possible to choose $c_{n,-} = 1$ and $c_{n,+} = b_n$. Usually bound states wave functions are labelled in increasing order, ψ_0 representing the ground state solution. It is possible to prove that ψ_n passes through zero exatly n times. This means that $b_n = |b_n|(-1)^n$. The set $s = \{r(k), \chi_n, |b_n|, n = 0, 1, \dots, \}$ is called *scattering data* and the mappping $V(x) \mapsto s$ is called *direct problem*. The *inverse problem* is the inverse mapping and has its own treatment. We have the following important result,

Theorem 2. *The following holds true:*

1. *$a(k)$ is analytical in the upper half-plane of k and $a(k) = 1 + O(1/k)$ as $|k| \to \infty$.*
2. *There is a one to one correspondence between bound state energies and the zeros of $a(k)$. Moreover the zeros lie on the imaginary axis, that is $a(i\chi_n) = 0$.*

This is the most important result and form the basis for a theory S-matrix based on analyticity. The theorem is easy to prove and we shall report the proof. The first step is to prove that $\phi(x,k)$ is analytical in the upper half-plane and satisfies the asymptotic behavior,

$$f_+(x,k) = e^{ikx}\phi(x,k) = 1 + O\left(\frac{1}{k}\right) \quad |k| \to \infty, \quad \Im(k) > 0 \tag{A.18}$$

where $\Im$ is the imaginary part. Analogously, $\psi(x,k)$ is analytical in the lower half-plane ($\bar{\psi}(x,\bar{k})$ analytical in the upper half) and $f_-(x,k) = e^{ikx}\psi(x,k)$ satisfies the same condition of f_+ for $\Im(k) < 0$. Second, calculating the Wronskians $W(\phi,\bar{\psi})$ in combination with (A.11) it is easy to see that,

$$a(k) = (2ik)^{-1}\left[\partial_x\bar{\psi}(x,k)\phi(x,k) - \bar{\psi}(x,k)\partial_x\psi(x,k)\right] = \frac{W(\phi,\bar{\psi})}{W(\psi,\bar{\psi})} \tag{A.19}$$

Thus, from analyticity properties of ψ and ϕ it follows that $a(k)$ is analytical in the upper half-plane $\Im(k) > 0$. Indeed we can represent the solution of the Schrödinger equation

as,

$$\phi(x,k) = e^{-ikx} - \int_{-\infty}^{+\infty} G(x,x',k)V(x')\phi(x,k)\mathrm{d}x' \tag{A.20}$$

where G is the appropriate Green's function for these boundary conditions. It is not difficult to compute G and it is found to be,

$$G(x,x',k) = \begin{cases} \frac{-\sin k(x-x')}{k} & x > x' \\ 0 & x < x' \end{cases} \tag{A.21}$$

This means that,

$$f_+(x,k) = 1 + \int_{-\infty}^{x} \frac{e^{2ik(x-x')}-1}{2ik} V(x')f_+(x',k)\mathrm{d}x' \tag{A.22}$$

This expression shows that f_+ can be analytically continued into the upper halpf-plane of k and that as $|k| \to +\infty$ has the correct asymptotic behavior. The same arguments lead to the analyticity of f_- in the lower half-plane. The first part of the theorem follows analytically continuing $a(k)$ into the upper half plane. The last part is to see the correspondence between bound states and zeros. If $a(k_0) = 0$ also $W(\phi(x,k), \bar{\psi}(x,\bar{k})|_{k=k_0} = 0$ and the functions $\phi(x,k_0)$, $\bar{\psi}(x,\bar{k}_0)$ and linearly dependent. The only possibility is,

$$\phi(x,k_0) = c\psi(x,\bar{k}_0) \tag{A.23}$$

This means that $\phi(x,k)$ vanishes both at $x \to +\infty$, by definition of ϕ, and at $x \to -\infty$ because $\bar{\psi}(x,\bar{k}_0) \sim e^{i\bar{k}_0 x}$ in this limit. Thus, $\phi(x,k_0)$ is a bound state with eigenvalue k_0^2. From the self-conugation condition k_0^2 is real that is the zeros of $a(k)$ lie on the imaginary axis. Conversely, if $E_n = -\chi_n^2$ is an eigenvalue of the problem, the function $\phi(x,i\chi_n)$ has the asymptotic behavior of a bound state as $x \to -\infty$. Thus,

$$\phi(x,i\chi_n) = b_n\bar{\psi}(x,-i\chi_n) = b_n\psi(x,-i\chi_n) \tag{A.24}$$

where we have used the fact the on the imaginary axis ψ is real valued. This complete the correspondence between bound states and zeros of a in the upper half-plane. Differentiating $(-\partial_x^2 + k^2)\phi(x,k) = 0$ with respect to k at $k = i\chi_n$ we find ,

$$(-\partial_x^2 + \chi_n^2)\frac{\mathrm{d}\phi(x,k)}{\mathrm{d}k}\bigg|_{k=i\chi_n} = 2i\chi_n\phi(x,i\chi_n) \tag{A.25}$$

Multiplying by $\phi(x,i\chi_n)$ and integrating over x gives,

$$\int_{-\infty}^{+\infty} \phi(x,i\chi_n)(-\partial_x^2 + \chi_n^2)\frac{\mathrm{d}\phi(x,k)}{\mathrm{d}k}\bigg|_{k=i\chi_n} \mathrm{d}x = 2i\chi_n \int_{-\infty}^{+\infty} \phi^2(x,i\chi_n)\mathrm{d}x \tag{A.26}$$

Integrating by parts twice the left hand side and expanding around $x \to +\infty$ the expression above,

$$\int_{-\infty}^{+\infty} \phi^2(x,k)\mathrm{d}x = i\frac{\mathrm{d}a(k)}{\mathrm{d}k}\bigg|_{k=i\chi_n} b_n \tag{A.27}$$

which proves that the zeros are also simple. This complete the discussion of the basic analytic properties of matrix T. We have used the transfer matrix T to go from one basis to another one. Waves of this basis correspond to waves on the left ($x \to -\infty$) and on the right ($x \to +\infty$). Usually in scattering experiments we control the *incoming waves* and measure the *outgoing* ones. Thus, it is of more interest, from a physical point of view, to have the linear transormation connecting in and outgoing waves. This is accompliched precisely by the S-matrix. Indeed, in our notations, ψ and $\bar{\phi}$ are the outgoing waves (we consider the target located at $x = 0$), while $\bar{\psi}$ and ϕ are ingoing waves. It is a matter of linear algebra to show that, the S-matrix is given by,

$$S(k) = \begin{pmatrix} -\frac{b(k)}{a(k)} & \frac{1}{a(k)} \\ \frac{1}{a(k)} & \frac{\bar{b}(k)}{a(k)} \end{pmatrix} \tag{A.28}$$

and acts as,

$$\begin{pmatrix} \psi \\ \bar{\phi} \end{pmatrix} = S \begin{pmatrix} \bar{\psi} \\ \phi \end{pmatrix} \tag{A.29}$$

We already see that the analytical structure of the scattering matrix is very complicated, even in the case of the Schröringer problem for a single degree of freedom. In relativistic field theories the LSZ reduction formula directly allows one to compute S-matrix elements perturbatively in the coupling parameter [89]. From the analytical properties of the coefficient a it is clear that bound states are to be sought at the simple poles of the S-matrix. Due to the non simple behavior of the function b it may have multiple branch points. While the T matrix has a special role in one dimensional problems, the S-matrix is more suited to construct general theories. This is because the symmetries of the Hamiltonian are explicit on the S-matrix. As an example, since probability must be conserved $S = S^\dagger$. In Chapter 3 we will use simlar ideas but within a more general formalism.

A.2 Analytic Scattering Theory

An alternative approach to scattering theory is that of analytic scattering theory, leading to the *bootstrap equations* [29, 90, 91]. For a pedagoginal introduction see Ref. [92]. We will not fill the details of this beautiful subject but only state the main results to give an idea of how profoundly different is physics in one spatial dimension. In practice, in the analytic theory of S-matrix one assumes a set of properties it should satisfy and a write down a set of consisency equations which fully determine its exact form once for all. These properties can be summarized as,

1. Interactions have to be short ranged

2. Superposition principle of quantum mechanics

3. Unitarity
4. Poincarè invariance
5. Causality principle
6. Analyticity principle

In this respect, the celebrated Coleman-Mandula theorem [56] states that in $d = 4$ if there is even a conserved charge transforming as tensor of rank $s \geq 2$ the S-matrix is trivial. In general, starting from a certain state in the distant past, scattering processes can give rise to arbitrary states in the distant feauture. This is not possible in one dimension as we will argue. If one considers a scalar bosonic theory in $1+1$ dimensions with ϕ^4 interaction one soon discovers that the *tree level amplitude* $2 \to 4$ processes is a constant. Adding a term ϕ^6 we can choose parameters to make this amplitude vanish. This time we discover that $2 \to 6$ amplitude a tree level is a constant. Adding more and more terms we end up with the following model, the Sinh-Gordon model,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{g^2}\left[\cosh(g\phi) - 1\right] \tag{A.30}$$

which has vanishing tree level amplitude for any $2 \to m$ process. Further, it is possible to prove that the one loop amplitude for this process vanishes. Thus, one is lead to think that such model could be special in some way, as particle production seems to be forbidden. Actually, if we consider $3 \to 3$ processes we will find a non zero amplitude. This is important since two things seem odd in this respect. First, we have a model for which the $2 \to 4$ amplitude is not found by crossing the $3 \to 3$ amplitude. Second, doing the calculation explicitely one can see also that the amplitude will not be an analytic function of the remaining momenta. This is in contrast with analyticity properties of S-matrix. In the main text we have discussed quantum integrability and we have talked about a definition of integrability based on conserved quantities. Even if this definition was not satisfactory, it is generally true that when a model is believed to be integrable it possess an infinite set of conservation laws in involution, mirroring the classical case, and conservation laws impose strong constraints on the structure of the theory. Of course the Coleman-Manudla theorem is out of scope so that we are safe in low dimension. It turns out that integrable theories have a special class of conserved charges, namely they are *local*, that is are expressible as an integral over a local density,

$$Q_s = \int q_s(x)\mathrm{d}x \tag{A.31}$$

These local charges are clearly operators acting on the Hilbert space of the theory. Usually, one assumes that this Hilbert space is specified once we give an Hamiltonin or a Lagrangian. Analytic theory of S-matrix was born also to find an alternative theory to quantum field theory [29] so that in this case no Hamiltonian or Lagrangian is specified. The Hilbert space is constructed out the asymptotic multiparticle states,

$$|A_{a_1}(\theta_1)\dots A_{a_n}(\theta_n)\rangle \tag{A.32}$$

where a_i label particles type and θ_i are the rapidities. In the literature of relativistic $1+1$ dimensional model it is customary to parametrize states by the rapidities of the

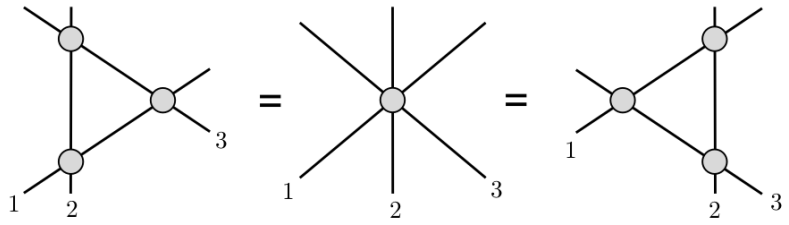

FIGURE A.2: Illustration of the S-matrix factorization in multiparticle processes for $1+1$ dimensional integrable theories.

particles: if $p = (p^0, p^1)$ is the 2-momentum,

$$p^0 = m\cosh(\theta) \quad p^1 = m\sinh(\theta) \tag{A.33}$$

Further, one defines the light cone coordinates as,

$$p = p^0 + p^1 = me^{\theta} \quad \bar{p} = p^0 - p^1 = me^{-\theta} \tag{A.34}$$

It is easy to recast the conservation of the 2-momentum in term of these variables in the light cone coordinates. Local charges with Lorentz spin s, being local, act additively on multiparticle states as,

$$Q_s \left|A_{a_1}(\theta_1)\dots A_{a_n}(\theta_n)\right\rangle = (q_{a_1}^{(s)} e^{s\theta_1} + \dots q_{a_n}^{(s)} e^{s\theta_n}) \left|A_{a_1}(\theta_1)\dots A_{a_n}(\theta_n)\right\rangle \tag{A.35}$$

Since these numbers must be conserved in any scattering process, the existence of local charges is sufficient to show that,

1. There is no particle production
2. Initial and final momenta are equal
3. Imporantly, the $n \to n$ S-matrix factorizes into a product of $2 \to 2$ scattering matrices, see Fig. A.2.

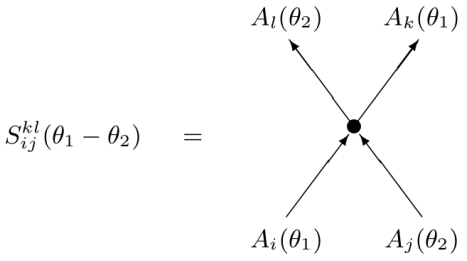

FIGURE A.3: Illustration of the S-matrix of the $2 \to 2$ process.

A pictorial representation of the 2-body Smatrix is given in Fig. A.3. Crossing invariance and unitarity in the same notation reads,

$$S_{ij}^{nm}(\theta) S_{nm}^{lk}(-\theta) = \delta_i^k \delta_j^l \tag{A.36}$$

$$S_{ij}^{kl}(\theta) = S_{ij}^{\bar{kl}}(i\pi - \theta) \tag{A.37}$$

where the bar denots antiparticle index.

In addition to these constraints on the elements of the scattering matrix there are the famous *Yang-Baxter* equations. Indeed, an analysis of the various possibilities of $3 \to 3$ processes shows that consistency requires the following conditions,

$$S_{ij}^{\beta\alpha}(\theta_{12})S_{\beta k}^{n\gamma}(\theta_{13})S_{\alpha\gamma}^{ml}(\theta_{23}) = S_{jk}^{\beta\gamma}(\theta_{23})S_{i\gamma}^{\alpha l}(\theta_{13})S_{\alpha\beta}^{nm}(\theta_{12}) \tag{A.38}$$

Certainly, all of what we said do not cover or explain many interesting aspects of the analytic theory of S-matrix for integrable systems such as the pole and branch cut structure deducible from its symmetry properties, but it is for sure enough to catch the main message. Scattering in $d = 2$ is strongly different, as particles are strongly interacting: there is no perturbative regime. The presence of local conservation laws limit the possibilities of processes and the knowledge of the 2-body scattering matrix is sufficient to determine all amplitudes. Further, internal consistencty, in many instances, allows to exactly compute this matrix based only on symmetry principles without specifying an underlying Lagrangian theory.

Appendix B

A Comment on Quantum Integrability

We have seen that in classical mechanics the defintion of integrability is a precise statement. If there is an equal number of independent conserved quantities in involution the Liouville's theorem ensures the problem is solved by quadrature. From the proof of the theorem, it is also true that the analytical problem of integrating a coupled set of differential equations has been transformed into a geometric problem of constructing the change of variables which trivializes the dynamics. The symplectic structure of the phase space descending from the introduction of the Poisson brackets plays a fundamental role in the achievement of this goal. In the quantum setting, however, the situation is quite different. Canonical variables, (p, q), which classically are functions of time, under quantization are promoted to operators acting on an Hilbert space, satisfing $[q, p] = i\hbar$. Now, even for a system like a single harmonic oscillator, the Hilbert space turns out to be infinite dimensional. Indeed, eigenfunctions of the hamiltonian are Hermite functions, $H_n(x)$. The problem in transposing the classical definition of integrability to the quantum case seems to be rooted in the way we count degrees of freedom. Indeed, classically we count the number of (q, p) pairs and this, quantum mechanically, would correspond to the way we count multiplicity of the infinite dimensionality of the Hilbert space: two harmonic oscillators would correspond to ∞^2 degrees of freedom. Even if we consider the more comfortable case of a quantum system with a finite dimensional Hilbert space, namely a single spin system does not make the situation better. That is because in quantum mechanics a fundamental notion is playes by a CSCO (complete set of commuting observables), meaning a set of commuting hermitian operators whose spectrums union fully describe the quantum system. For a full non-degenerate system, a single operator, namely the Hamiltonian, already forms a CSCO (take the powers H^n). Thus, the attention may go the a maximal Abelian subalgebra of observables, (realtives of the Cartan subalgebra in Lie theory) whose number equals the dimensionality of the Hilber space. Having said so, it is clear that the word "complete", the most used in the literature, is rather unfortunate and should be substituted with "maximal". One of the most used definitions used in the literature is that a quantum system is integrable if it possess a maximal set of commuting operators Q_α, $\alpha = 1, \ldots, \dim(\mathcal{H})$. This is the literal translation of the classical deifintion of Liouville. An immediate reason why this is not satisfactory is that in any quantum system it is possible to define projectors $Q_\alpha = |\psi_\alpha\rangle \langle\psi_\alpha|$ and these constitute a maximal set of commuting operators. It would follow that any theory is integrable. More importantly, there exists a theorem by Von Neumann [93] hermitian operators into a single function $Q_\alpha = f_\alpha(\mathbf{Q})$, so the notion of the number of independent operators seems ill-defined. It is at this point clear the the above naive definition of integrability is far from being acceptable. Other notions of integrablity have been proposed but the most important seems the one rooted on the physics of integrable models.

(quasi-)Definition 1. *(Quantum Integrability) A quantum system is integrable if the scattering it supports is non-diffractive (no particle production, totally elastic).*

The scattering in integrable models displays particular characteristic: the S-matrix in integrable theories can be factored in two-body S-matrices and must satisfy a particular set of stringent equations. The Bethe Ansatz solution of many integrable models provides a visible occurrence of this feature. Recently, a more complicated definition of quantum integrability has been attempted in Ref. [16]. One basic requirement the authors look for in a good definition is that it should be able to classify models into different "integability classes". In this respect, the definition mimics in some sense the one given in complexity theory to classify problems according the time needed to their solution. Thus, there will be linear, sub-linear and polynomial integrability theories. For more details the reader is addressed to the literature.

Appendix C

Virial Identities

The Virial Theorem in classical mechanics simply says the the time average of a total derivative vanishes.

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t)\mathrm{d}t=0$$

This statement can be used, in combination with field equations, to generate a set of non trivial identities between the field and other observables. For example consider the complex field of the NLSE dimensionless Hamiltonian. The quantity,

$$I(t)=\frac{1}{2}\int_0^L\mathrm{d}x\Psi \tag{C.1}$$

gives,

$$\dot{I}(t)=\int_0^L\mathrm{d}x\dot{\Psi}=-i\int_0^L\mathrm{d}x\left(-\partial_x^2\Psi+2|\Psi|^2\Psi\right) \tag{C.2}$$

Dividing by L and taking the time average we get,

$$\langle\partial_x^2\Psi\rangle_{CDA}=2\langle|\Psi|^2\Psi\rangle_{CDA} \tag{C.3}$$

showing that in the NLSE at large times the contribution of the two terms kinetic and interaction terms is perfectly balanced (see Fig. 2.4). It is evident tha varying the quantity I one can derive an infinite tower of these identities linking different expectation values. Working with the complex field is cumbersome because our potential explicitely mix real and imaginary part. To see an illuminating and non trivial virial identity we can consider for a moment the relativisti Lagrangian (2.53) and the quantity,

$$I(t)=\int_0^L\mathrm{d}x\varphi^2 \tag{C.4}$$

Deriving with respect to time twice,

$$\ddot{I}(t)=\int_0^L\mathrm{d}x\left\{\varphi\ddot{\varphi}+(\dot{\varphi})^2\right\} \tag{C.5}$$

Using again field equations and taking the time average we find,

$$\langle\dot{\varphi}^2\rangle_{CDA}=\langle\partial_x\varphi^2\rangle_{CDA}+\langle\varphi\frac{\mathrm{d}V_R}{\mathrm{d}\varphi}\rangle_{CDA} \tag{C.6}$$

From this we get the energy per unit length,

$$E/L = \langle \dot{\varphi}^2 \rangle_{CDA} + \langle V_R - \varphi \frac{\mathrm{d}V_R}{\mathrm{d}\varphi} \rangle_{CDA} = \langle (\partial_x \varphi)^2 \rangle_{CDA} + \langle V_R + \varphi \frac{\mathrm{d}V_R}{\mathrm{d}\varphi} \rangle_{CDA} \tag{C.7}$$

this expression can be used to fix the temperature in relativistic field theories in terms of the energy per degree of freedom. It is also interesting to compare the relativistic expression for the energy per unit length with its non relativistic counterpart,

$$E/L = \langle \frac{1}{2m} |\partial_x \Psi|^2 \rangle_{CDA} + \langle V_{NR} \rangle_{CDA} \tag{C.8}$$

In the relativistic theory a sort of effective potential contribution arise.

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